M 382D: Differential Topology
Spring 2015

Exercise Set 2
Due: Fri Feb 6

Exercise 1. Guillemin/Pollack Chapter 1, §2 (p. 11): 2, 4, 10, 11

Exercise 2. Consider the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{A}^3 : x^2 + y^2 + z^2 = 1\}.$$

- There is an obvious inclusion $i : S^2 \to \mathbb{A}^3$. Show that the differential di_p at any point $p \in S^2$ is an injection $di_p : T_pS^2 \to \mathbb{R}^3$ and identify the image.
- On the upper hemisphere $U = \{z > 0\}$ the functions (x, y) give a chart. To get a second chart take spherical coordinates (θ, ϕ), related to (x, y) by

$$\begin{aligned} x &= \sin\phi\cos\theta \\ y &= \sin\phi\sin\theta. \end{aligned}$$

  Identify some (maximal) subset of U on which (θ, ϕ) give a chart. At any point p on that subset, express the vector $\partial/\partial x \in T_pS^2$ in terms of $\partial/\partial\theta \in T_pS^2$ and $\partial/\partial\phi \in T_pS^2$. Also, compute $di(\partial/\partial x) \in T_{i(p)}\mathbb{A}^3$.

Exercise 3. Fix positive numbers r and R with $r < R$. Let the torus T be the surface of revolution in $\mathbb{A}^3$ (with coordinates x, y, z) obtained by revolving the circle

$$y = 0, \quad (x - R)^2 + z^2 = r^2$$

about the z-axis.

- Show that T is a 2-manifold.
- Define the *Gauss map* $g : T \to S^2$ by mapping a point $p \in T$ to the unit normal vector to T at p, considered as a point of S^2. (Here rely on the notion of "unit normal" you have studied before; we have not discussed it in this class.) Show that g is smooth. Compute its differential in some coordinate system.

Exercise 4. Let $P(z)$ be a polynomial in a single complex variable. Consider the family of equations $P(z) = s$ for a variable complex number s. Suppose that for some z_0, s_0 we have $P(z_0) = s_0$, and z_0 is a *simple* root of $P(z) - s_0$. Let $t \mapsto s_t$ be a smooth curve through s_0. Prove that there is a smooth curve $t \mapsto z_t$, for t in a neighborhood of 0, such that $P(z_t) = s_t$. What happens at t for which $P(z) - s_t$ develops a double root?

Exercise 5. This problem gives a standard and important corollary of the inverse function theorem. It states a condition under which we can solve an equation of two variables implicitly for one variable as a function of the other.

Suppose X, Y, Z are manifolds and $F : X \times Y \to Z$ a smooth map with $F(x_0, y_0) = z_0$ for some $x_0 \in X$, $y_0 \in Y$, and $z_0 \in Z$. Assume that the restriction of the differential $dF_{(x_0,y_0)}$ to $T_{y_0}Y \subset T_{(x_0,y_0)}(X \times Y)$ is an isomorphism onto $T_{z_0}Z$. Prove that there exists a neighborhood U of x_0 and V of y_0 and a smooth function $f : U \to V$ such that

$$F\big(x, f(x)\big) = z_0$$

for all $x \in U$. (More generally, for z in a neighborhood of z_0, we can find a function $f_z : U \to V$ which solves the equation $F\big(x, f_z(x)\big) = z$.)

Exercise 6. Let X be a complete metric space and S any metric space. Suppose

$$K : S \times X \to X$$

is a continuous map and a contraction in $x \in X$ uniformly over $s \in S$, i.e., there is a constant $0 < C < 1$ such that

$$d\big(K(s, x_1), K(s, x_2)\big) \leq C d(x_1, x_2), \qquad s \in S, \quad x_1, x_2 \in X.$$

Let $x_s \in X$ denote the fixed point of $K(s, -) : X \to X$. Prove that the map $S \to X$ given by $s \mapsto x_s$ is continuous.