M 382D: Differential Topology
Spring 2015

Exercise Set 3
Due: Mon Feb 16

Exercise 1. Guillemin/Pollack Chapter 1, §4 (p. 25): 2, 5, 9, 10. For 9 and 10 you need the definition of the orthogonal group: it is the group of all $n \times n$ real matrices A obeying $AA^T = 1$ (as discussed on pages 22-23).

Exercise 2. This exercise is essentially a tautology, which we already mentioned in class, but which seems worth a moment's reflection.

Let M be a smooth manifold, $U \subset M$ an open subset, and $x : U \to \mathbb{A}^n$. Prove that (U, x) is a chart if and only if $x : U \to f(U)$ is a diffeomorphism.

Exercise 3. Let M be a manifold. The *tangent bundle* TM is defined as the disjoint union of all the tangent spaces, $\bigcup_{p \in M} T_pM$. Equip TM with a natural topology and smooth atlas. (Hint: Consider a chart (U, x) of M. Then, essentially by our *definition* of T_pM, we have a natural identification between the open subset $\bigcup_{p \in U} T_pM \subset TM$ and $U \times \mathbb{R}^n$.) What is the dimension of TM? Can you describe TS^1 as a familiar manifold?

Exercise 4. This exercise gives a little more practice working with projective spaces.

1. Define complex projective space $\mathbb{CP}^n$ as the set of equivalence classes
$$\mathbb{CP}^n = \{[z^0, z^1, \dots, z^n] : (z^0, z^1, \dots, z^n) \neq (0, 0, \dots, 0)\} /{\sim},$$
where
$$[z^0, \dots, z^n] \sim [z'^0, \dots, z'^n] \qquad \text{if and only if} \qquad z'^i = \lambda z^i$$
for some $\lambda \in \mathbb{C}^\times$. Put a natural structure of smooth manifold on $\mathbb{CP}^n$. (Consider $U_i = \{[z^0, \dots, z^n] : z^i \neq 0\}$.) Construct a diffeomorphism between $\mathbb{CP}^1$ and the standard 2-sphere.

2. Suppose you identify the 3-sphere with the unit sphere in $\mathbb{C}^2$
$$S^3 = \{(z^1, z^2) \in \mathbb{C}^2 : |z^1|^2 + |z^2|^2 = 1\}.$$
Then show that the map
$$\begin{aligned} f : S^3 &\longrightarrow S^2 \\ (z^1, z^2) &\longmapsto [z^1, z^2] \end{aligned}$$
is a submersion. What is the inverse image of a point? The map f is called the *Hopf fibration*. (For fun, think about what you can say about the inverse image of a 2-point subset of S^2.)

Exercise 5. Here is a more abstract, coordinate-free approach to the manifold structures on projective spaces. Let V be a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ and denote by $\mathbb{P}V$ the set of lines in V. Recall that a *line* is a 1-dimensional vector space, so a line in V is a one-dimensional subspace of V.

Let $L \subset V$ be a line and $W \subset V$ a complementary subspace, i.e., $V = L \oplus W$. Define

$$\begin{aligned} \phi_{L,W} : \mathrm{Hom}(L,W) &\longrightarrow \mathbb{P}V \\ T &\longmapsto L_T \end{aligned}$$

where $L_T = \{\ell + T\ell : \ell \in L\}$ is the graph of T. We can identify L_T as the image of the linear map $\mathbf{1}_L + T : L \to L \oplus W = V$. Show that the image of $\phi_{L,W}$ is $\mathbb{P}V \setminus \mathbb{P}W$.

Now consider a second pair (L', W') and the corresponding $\phi_{L',W'}$. We now have two parametrizations of $\mathbb{P}V \setminus (\mathbb{P}W \cup \mathbb{P}W')$, so can compare by an overlap isomorphism

$$f : U \longrightarrow U',$$

where $U \subset \mathrm{Hom}(L,W)$ and $U' \subset \mathrm{Hom}(L',W')$ are the images of $\mathbb{P}V \setminus (\mathbb{P}W \cup \mathbb{P}W')$ under the two parametrizations. Write a formula for the map f. Show that f is smooth. (Hint: The formula involves $\pi^{L',W'} : V \to L'$, the projection onto L' with kernel W'.)

Use the parametrizations $\phi_{L,W}$ to topologize $\mathbb{P}V$ and construct a smooth atlas, so make $\mathbb{P}V$ a smooth manifold. What is its dimension? Prove that an injective linear map $V' \to V$ induces a smooth map $\mathbb{P}V' \to \mathbb{P}V$.

Construct a natural isomorphism (take this to mean that it doesn't depend on choices)

$$T_L(\mathbb{P}V) \to \mathrm{Hom}(L, V/L).$$

Exercise 6. This exercise is preparation for our discussion of partitions of unity.

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} e^{-1/x^2}, & x > 0; \\ 0, & x \leq 0. \end{cases}$$

   Prove that f is C^∞. Sketch the graph of f. Compare f to its Taylor series at $x = 0$.

2. Given real numbers $a < b$ show that

   $$g(x) := f(x-a)f(b-x)$$

   is smooth and vanishes outside the interval (a,b).

3. Given real numbers $a < b$, construct a C^∞ function h such that: (i) $h(x) = 0$ for $x \leq a$, (ii) $h(x) = 1$ for $x \geq b$, and (iii) h is monotonic nondecreasing.

4. Given real numbers $a < b < c < d$, construct a C^∞ function k so that (i) $k(x) = 0$ for $x \leq a$, (ii) $k(x) = 1$ for $b \leq x \leq c$, and (iii) $k(x) = 0$ for $x \geq d$.

5. Given real numbers $a^i < b^i < c^i < d^i$, $i = 1, \dots, n$, construct a C^∞ function $k : \mathbb{A}^n \to \mathbb{R}$ so that (i) $k(x^1, \dots, x^n) = 0$ if any $x^i \leq a^i$; (ii) $k(x^1, \dots, x^n) = 1$ if $b^i \leq x^i \leq c^i$ for all $i = 1, \dots, n$; and (iii) $k(x^1, \dots, x^n) = 0$ if any $x^i \geq d^i$.

6. Prove that on every manifold M there is a nonconstant C^∞ function $f : M \to \mathbb{R}$.