M 382D: Differential Topology
Spring 2015

Exercise Set 4
Due: Mon Feb 23

Exercise 1. Guillemin/Pollack: Chapter 1, §7 (p. 45): 4, 6 (assume f is smooth)

Exercise 2. Guillemin/Pollack: Chapter 1, §8 (p. 55): 3

Exercise 3. Prove that if $f : M \to N$ is a smooth map, then the differential $df : TM \to TN$ is also a smooth map.

Exercise 4. Recall that real projective space $\mathbb{RP}^n$ may be defined as the set of nonzero real $(n+1)$-tuples $x = (x^0, x^1, \ldots, x^n)$ up to an equivalence which identifies two $(n+1)$-tuples if one is obtained from the other using scalar multiplication by a nonzero constant.

1. Let $F = F(x^0, \ldots, x^n)$ be a homogeneous real-valued function: $F(\lambda x) = \lambda^r F(x)$ for some real number r and all nonzero $\lambda \in \mathbb{R}$. How does the equation $F = 0$ define a subset of $\mathbb{RP}^n$?

2. What condition on F guarantees that this subset is a submanifold?

3. Homogeneous polynomials are particular examples of homogeneous functions. Show that any linear polynomial F satisfies the condition you found in the previous part. What is the corresponding submanifold of $\mathbb{RP}^n$?

4. Now investigate (homogeneous) quadratic and cubic polynomials. You might try the case $n = 2$ first to see what sort of submanifolds you get.

Exercise 5. For each of the following construct an example.

1. A compact manifold M and a smooth manifold N with $\dim M = \dim N = 2$, and a smooth map $f : M \to N$ such that if $R \subset N$ is the subset of regular values and $\# : R \to \mathbb{Z}^{\geq 0}$ the function which assigns to $q \in R$ the cardinality of $f^{-1}(q)$, then # takes on three distinct values. (Recall from lecture that # is locally constant.)

2. An embedding $f : M \to N$ which is not proper.

3. A non-simply connected compact 4-manifold.

4. A surjective local diffeomorphism of 3-manifolds which is not a diffeomorphism.