M 382D: Differential Topology
Spring 2015

Exercise Set 6
Due: Wed Mar 11

Exercise 1. Guillemin/Pollack: Chapter 2, §2 (p. 66): 2, 3, 6, 7

Exercise 2. This exercise concerns the construction of vector bundles via "gluing data."

1. Suppose given a smooth vector bundle E over a smooth manifold M. Then by definition we have an open cover of M by patches U_α, and for each patch we have a smooth "local trivialization" $\phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^k$, taking fibers to fibers. Show that the overlap maps $\psi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ can be viewed as smooth maps $U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$, that on triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$ they obey $\psi_{\alpha\beta} \circ \psi_{\beta\gamma} \circ \psi_{\gamma\alpha} = 1$, and that $\psi_{\alpha\alpha} = 1$.

2. Conversely, suppose given an open cover of M by patches U_α, together with a collection of smooth maps $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$, obeying $\psi_{\alpha\beta} \circ \psi_{\beta\gamma} \circ \psi_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$, and $\psi_{\alpha\alpha} = 1$. Then, construct a vector bundle E such that the $\psi_{\alpha\beta}$ are the overlap maps. (One way to define E is by "gluing together": begin with the disjoint union of the sets $U_\alpha \times \mathbb{R}^k$ and then impose an appropriate equivalence relation. Then you have to show that the resulting E indeed has the structure of a vector bundle. For its topology you should take the quotient topology — this is parallel to the email I sent you a few weeks ago concerning gluing constructions of manifolds.)

3. Show that any vector bundle E can be obtained by the kind of gluing construction you described in the previous part.

4. Now suppose given a vector bundle E obtained by the gluing construction, with overlap maps $\psi_{\alpha\beta}$. Show that the overlap maps $\tilde{\psi}_{\alpha\beta} = (\psi^T)^{-1}_{\alpha\beta}$ also obey the triple overlap condition. Thus we can define another vector bundle $\tilde{E}$ with overlap maps $\tilde{\psi}_{\alpha\beta}$. Show that in fact $\tilde{E}$ is the dual bundle E^* which we defined in class.

5. In a similar spirit, given two vector bundles E and F, explain how to construct vector bundles $E \oplus F$ and $E \otimes F$, with the property that $(E \oplus F)_p = E_p \oplus F_p$ and similarly for $\otimes$. You can do this in any way you want, but one convenient way is to use the gluing construction: then the question is just to describe the gluing maps for $E \oplus F$ and $E \otimes F$ in terms of those for E and F.

Exercise 3. Give an example of:

1. a vector bundle E which is not globally trivial,

2. a vector bundle E which is not globally trivial, but which admits a global section $s \in \Gamma(E)$ that does not vanish anywhere.

Exercise 4. Show that if N is a neat submanifold of a manifold-with-boundary M then N intersects ∂M transversely.