M 382D: Differential Topology
Spring 2015

Exercise Set 7
Due: Wed Mar 25

Exercise 1. Give examples of the following.

1. A compact manifold M such that there does not exist a compact manifold N with boundary such that ∂N is diffeomorphic to M.

2. A connected noncompact manifold M with boundary such that ∂M is compact and connected.

3. A connected noncompact manifold M with boundary such that ∂M is noncompact and connected.

Exercise 2. This exercise constructs splittings of exact sequences of vector bundles. (In class we explained how to do this in a "geometric" way for the particular exact sequence defining the normal bundle to a submanifold.)

1. Suppose
$$0 \longrightarrow E' \xrightarrow{\;i\;} E \xrightarrow{\;j\;} E'' \longrightarrow 0$$
is a short exact sequence of vector spaces. Recall this means i is injective, j is surjective, and $\ker j = i(E')$. A *splitting* is a linear map $s : E'' \to E$ which is right inverse to j, i.e. $j \circ s = 1$. Prove that the space of splittings is an affine space over the vector space $\operatorname{Hom}(E'', E')$. Prove also that a splitting is equivalent to a left inverse to i.

2. Let A be a real affine space and $a_1, \dots, a_n \in A$. Suppose $\rho_1, \dots, \rho_n \in \mathbb{R}$ satisfy $\rho_1 + \cdots + \rho_n = 1$. Make sense of the element $\rho_1 a_1 + \cdots + \rho_n a_n \in A$.

3. Now suppose M is a smooth manifold and
$$0 \longrightarrow E' \xrightarrow{\;i\;} E \xrightarrow{\;j\;} E'' \longrightarrow 0$$
a short exact sequence of vector bundles over M. Use a partition of unity argument to construct a splitting of this sequence.

Exercise 3. The *quaternions* $\mathbb{H}$ are a division algebra over $\mathbb{R}$ which is a 4-dimensional vector space with basis $1, i, j, k$ and multiplication rule $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Construct a norm on the quaternions by setting $|a + bi + cj + dk|^2 = a^2 + b^2 + c^2 + d^2$, where $a, b, c, d \in \mathbb{R}$. Identify S^3 as the unit sphere in $\mathbb{H}$. Use the quaternion multiplication to construct a global trivialization of TS^3, that is, three vector fields on S^3 which for each $p \in S^3$ form a basis of T_pS^3. (Hint: Warm up by replacing $\mathbb{H}$ with $\mathbb{C}$ and S^3 with S^1.)

Exercise 4. Suppose M is a smooth manifold, $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}^k$. Show that $\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f$ (and explain what this equation means).