M 382D: Differential Topology
Spring 2015

Exercise Set 8
Due: Mon Apr 6

Exercise 1. Guillemin/Pollack (a lot, but many of them are very short): Chapter 2, §4 (p. 82): 3, 5, 6, 7, 9, 10 (hint: use intersection numbers), 11, 13

Exercise 2. For each of the following construct an example (with justification) or show that it does not exist.

1. A map $f : S^1 \times S^1 \to S^2$ with $\deg_2 f \neq 0$.
2. A map $f : S^2 \to S^1 \times S^1$ with $\deg_2 f \neq 0$.
3. A map $f : S^2 \to \mathbb{RP}^2$ with $\deg_2 f \neq 0$.
4. A map $f : \mathbb{RP}^2 \to S^2$ with $\deg_2 f \neq 0$.

Exercise 3. A *knot* is the image of an embedding $f : S^1 \to \mathbb{A}^3$. Suppose we have two disjoint knots, which are the images of maps $f, g : S^1 \to \mathbb{A}^3$. Define the mod 2 *linking number* as the mod 2 degree of the map

$$f \times g : S^1 \times S^1 \longrightarrow S^2$$
$$s \times t \longmapsto \frac{f(s) - g(t)}{|f(s) - g(t)|}$$

1. Compute the linking number between the unit circle in the x-y plane centered at $(0,0,0)$ and the unit circle in the y-z plane centered at $(0, \frac{1}{2}, 0)$.
2. Suppose that f extends to a map $F : D^2 \to \mathbb{A}^3$ with $\partial F = f$, where D^2 is the unit disk with boundary S^1. We may assume that F is transverse to $g(S^1)$. Prove that the linking number is the number of points in $F^{-1}\big(g(S^1)\big)$ mod 2.

Exercise 4. This exercise concerns the "linearity principle" that allows us to recognize when a functional on sections of vector bundles itself comes from a sections of a vector bundle.

1. Suppose E is a smooth vector bundle over a manifold M, and $\phi : \Gamma(E) \to \mathbb{R}$ is linear over $C^\infty(M)$, in the sense that $\phi(s + s') = \phi(s) + \phi(s')$ and for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$ we have
$$\phi(fs) = f\phi(s)$$
Then, show that there exists some $\eta \in \Gamma(E^*)$ such that
$$\phi(s) = \eta(s)$$
(Hint: use the existence of local trivializations for E, which implies that locally there exist sections $s_1, \ldots, s_k \in \Gamma(E)$ such that any section s can be expanded in the form $s = \sum_{i=1}^k f_i s_i$, with f_i smooth functions.)

2. Suppose given $\phi : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$ which is alternating and multilinear over $C^\infty(M)$. Show that ϕ factors through a canonical map

$$\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \Gamma(\wedge^k TM)$$

and thus we can view ϕ as a map $\phi : \Gamma(\wedge^k TM) \to C^\infty(M)$ which is linear over $C^\infty(M)$. Using the previous part, conclude that ϕ comes from some $\omega \in \Omega^k(M)$, as we claimed in class.

Exercise 5. For any $\alpha, \beta \in \Omega(N)$ and $f : M \to N$ smooth show that

$$f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta.$$

Exercise 6. Consider a 1-form $\alpha = g(x)\mathrm{d}x$ on the affine line $\mathbb{A}^1$. Prove that there exists a function $f(x)$ so that $\alpha = \mathrm{d}f$. Then try the same problem with $\mathbb{A}^1$ replaced by the circle S^1. (Or equivalently, replace α and f with a periodic 1-form and a periodic function.)