M 382D: Differential Topology
Spring 2015

Exercise Set 9

Due: Wed Apr 15 (tax day)

Exercises marked with $(\star)$ will definitely not be graded.

Exercise 1. Let $\phi : \mathbb{A}^3 \to \mathbb{A}^3$ be given by

$$(x^1, x^2, x^3) \mapsto (y^1, y^2, y^3) = (x^1x^2, x^1x^3, x^2x^3).$$

Compute $\phi^*(\mathrm{d}y^1)$, $\phi^*(\mathrm{d}y^1 \wedge \mathrm{d}y^2)$, $\phi^*(\mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3)$, and $\phi^*(y^1y^2\mathrm{d}y^3)$.

Exercise 2. This exercise gives a little practice with changes of coordinates for vector fields and forms. Our conventions for spherical coordinates on S^2 are

$$x = \cos\theta, \quad y = \sin\theta\sin\varphi, \quad z = \sin\theta\cos\varphi.$$

1. Show that there exists a smooth vector field $\xi \in \mathfrak{X}(S^2)$ such that, on every patch where spherical coordinates (θ, φ) give a diffeomorphism, we have $\xi = \frac{\partial}{\partial\varphi}$. (There are two things that have to be checked: one is that this vector field glues well along overlaps of patches, the other is that it extends smoothly to the poles of S^2, which are not covered by any spherical coordinate patch.)

2. Show in contrast that there is *no* smooth vector field $\xi \in \mathfrak{X}(S^2)$ which has $\xi = \frac{\partial}{\partial\theta}$ on every such patch.

3. Show that there is a 2-form $\omega \in \Omega^2(S^2)$ which has $\omega = \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi$ on each such patch. Show moreover that ω is nowhere vanishing on S^2. Why is this not a contradiction with the fact that $\sin\theta$ vanishes at $\theta = 0$?

4. Compute $\int_{S^2} \omega$. (You may freely use the principle mentioned but not proven in class, that we are allowed to excise subsets of measure zero.)

5. Show that ω of the previous part is not exact, i.e. there is no $\eta \in \Omega^1(S^2)$ such that $\omega = \mathrm{d}\eta$.

6. Show that the form $\mathrm{d}x$ on $\mathbb{A}^1$ does not extend to the one-point compactification S^1 (thought of as a smooth manifold), while the vector field $\frac{\partial}{\partial x}$ does extend. Show that the extended vector field vanishes at the added point $x = \infty$. (Hint: to save a little effort, you can actually deduce the fact that $\mathrm{d}x$ does not extend to $x = \infty$ from the fact that the extended vector field vanishes there.)

Exercise 3. In this problem you will study differential forms on Euclidean 3-space $\mathbb{E}^3$ and relate the exterior derivative d to div, grad, and curl. (Note $\mathbb{E}^3$ is setwise the same as $\mathbb{A}^3$; we call it $\mathbb{E}^3$ to emphasize the inner product on the underlying vector space; the relevance of this point will appear below.)

Suppose $\xi = P(x,y,z)\frac{\partial}{\partial x} + Q(x,y,z)\frac{\partial}{\partial y} + R(x,y,z)\frac{\partial}{\partial z}$ is a vector field on $\mathbb{E}^3$. We associate to ξ a 1-form α_ξ and a 2-form β_ξ by the formulas

$$\begin{aligned}\alpha_\xi &= P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z,\\ \beta_\xi &= P\mathrm{d}y \wedge \mathrm{d}z + Q\mathrm{d}z \wedge \mathrm{d}x + R\mathrm{d}x \wedge \mathrm{d}y.\end{aligned}$$

These formulas give isomorphisms

$$\mathfrak{X}(\mathbb{E}^3) \simeq \Omega^1(\mathbb{E}^3) \simeq \Omega^2(\mathbb{E}^3).$$

Also, we can associate a 3-form ω_f to a function f by the formula

$$\omega_f = f(x,y,z)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

1. These isomorphisms are made pointwise, so belong to linear algebra. That is, they are derived from similar isomorphisms for a 3-dimensional real inner product space. So, fix a 3-dimensional real inner product space V. Choose an orthonormal basis for V and define isomorphisms $V \simeq V^* \simeq \wedge^2 V^*$ by imitating the formulas above. Explain how these isomorphisms depend on the choice of basis (you should find that for two orthonormal bases which define the same orientation on V you get the same isomorphisms, but if the two bases define different orientations there are some extra signs.)

2. Identify the composition

$$\Omega^0(\mathbb{E}^3) \xrightarrow{\mathrm{d}} \Omega^1(\mathbb{E}^3) \to \mathfrak{X}(\mathbb{E}^3)$$

   with the operator taking the gradient of a function. (The second map is the isomorphism above.) Generalize to $\mathbb{E}^n$ for any n.

3. Identify the composition

$$\mathfrak{X}(\mathbb{E}^3) \to \Omega^1(\mathbb{E}^3) \xrightarrow{\mathrm{d}} \Omega^2(\mathbb{E}^3) \to \mathfrak{X}(\mathbb{E}^3)$$

   with the curl operator. (The first and last maps are the isomorphisms above.)

4. Identify the composition

$$\mathfrak{X}(\mathbb{E}^3) \to \Omega^2(\mathbb{E}^3) \xrightarrow{\mathrm{d}} \Omega^3(\mathbb{E}^3) \to \Omega^0(\mathbb{E}^3)$$

   with the divergence operator. (The first and last maps are the isomorphisms above.)

5. What is the meaning of the identity $\mathrm{d}^2 = 0$ in terms of divergence and curl?

6. ($\star$) Can you say anything about analogues in higher dimensions?

Exercise 4. Suppose V is a vector space. Let $or(V)$ denote the set of orientations of V; given $o \in or(V)$ let $-o \in or(V)$ denote the other orientation. Now define the space of *densities on* V, $D(V)$, to be

$$D(V) = (\det(V) \times or(V)) / \left[(\omega, o) \sim (-\omega, -o)\right].$$

1. ($\star$) Give a natural structure of 1-dimensional vector space on $D(V)$. Show that, given a map $T : V \to W$ with $\dim V = \dim W$, there is a natural induced map $D(T) : D(V) \to D(W)$, which behaves functorially, i.e. $D(T \circ T') = D(T) \circ D(T')$. Show that if $T : V \to V$ then the corresponding map $D(T) : D(V) \to D(V)$ is multiplication by $|\det T|$. The absolute value sign here is the crucial new point.

2. ($\star$) If M is a smooth manifold, construct a line bundle $D(M)$ such that $D(M)_p = D(T_p^* M)$: this is the bundle of *densities on* M. Show that $D(M)$ admits nonvanishing global sections, whether or not M is orientable (in contrast to $\det T^* M$ which admits nonvanishing global sections if and only if M is orientable).

3. ($\star$) For $\rho \in \Gamma(D(M))$ define $\int_M \rho$. The definition should be very similar to our definition of integration of differential forms over *oriented* M, but it should *not* require M to be oriented. A precise way of stating the relation between the two notions of integration is: if M does happen to be oriented (with orientation o), then given $\omega \in \Omega^m(M)$ we can define a corresponding $\rho \in \Gamma(D(M))$ by $\rho(p) = (\omega(p), o(p))$, and in that case we should have $\int_M \omega = \int_M \rho$.