What is differential topology?

topology: properties of topological spaces X, invariant under homeomorphism

differential topology: properties of smooth manifolds M, invariant under smooth maps

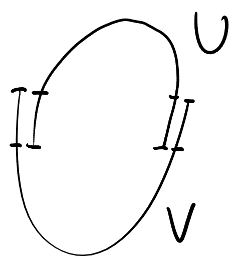
What is a smooth map?

We know the answer for maps $f: \overset{\mathbb{A}^n}{\overset{\cup}{U}} \longrightarrow \overset{\mathbb{A}^m}{\overset{\cup}{U'}}$ — call them smooth if they are infinitely differentiable.

[$\mathbb{A}^n = \{ (x^1, \dots, x^n) \mid x^i \in \mathbb{R} \}$]

Build smooth manifolds by "gluing together" pieces that look like $U \subset \mathbb{A}^n$.

[Ex



$\varphi: U_0 \xrightarrow{\sim} V_0$ "patching data"

gives the smooth manifold S^1 (circle)

Similarly build n-sphere S^n by patching two hemispheres.]

Then, can define smooth maps $f: M \to N$ and

- develop tools of calculus, e.g. what is the derivative of f?
  how do we integrate f? (A: we don't, we integrate differential forms / densities)
- use these tools to study f. Smoothness helps a lot! e.g.
  for $f: M \to N$ with $\dim(M) < \dim(N)$, $f(M)$ has measure zero
  (cf. space-filling curves)
  $f: M \to S^k$ with $\dim(M) = k$ (and M orientable) are classified "up to smooth homotopy" by a single invariant, $\deg f \in \mathbb{Z}$.
  $\deg f = \# f^{-1}(p) \bmod 2$ for almost every $p \in S^k$.
- use these tools to study M. Morse theory.