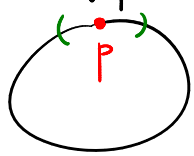
Def M a topological space:

1) M is <u>locally Euclidean</u> if, $\forall p \in M$, $\exists$ nbhd $U \subset M$ ($p \in U$) and a homeomorphism $U \xrightarrow{\sim} V$ where $V \subset \mathbb{A}^n$ open.

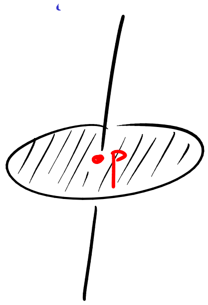
2) M is <u>topological manifold</u> if M is locally Euclidean, Hausdorff, and has a countable basis.

Ex $M = S^1$ is locally Euclidean: every point has a neighborhood homeomorphic to an open interval $V \subset \mathbb{A}^1$.



Also Hausdorff and has countable base.

Ex



union of disc and line is not locally Euclidean at p.

This follows from

Thm (Invariance of Domain) If $V_1 \subset \mathbb{A}^{n_1}$, $V_2 \subset \mathbb{A}^{n_2}$ both open and $V_1 \approx V_2$ (homeo) then $n_1 = n_2$.

Pf Not easy! Omitted here. (Standard way uses homology)

Cor If M is locally Euclidean, there is a <u>unique</u>, <u>locally constant</u> function $\dim_M : M \to \mathbb{Z}_{\geq 0}$ such that, for $p \in M$, $\exists$ nbhd U of p with $U \approx V \subset \mathbb{A}^{\dim_M(p)}$.

Pf Fix n, show $\dim_M^{-1}(n)$ <u>open</u> and <u>closed</u> in M.

Ex "Line with doubled origin" $\mathbb{A}^1 \cup_{\mathbb{A}^1 \setminus \{0\}} \mathbb{A}^1$

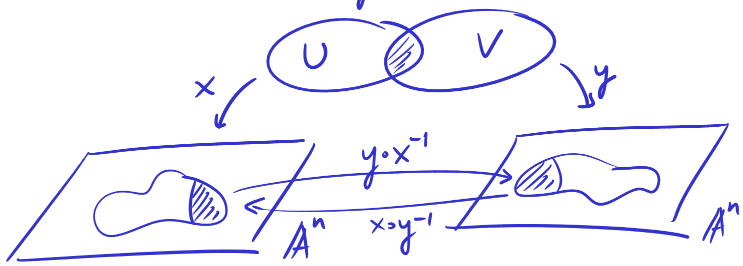
is locally Euclidean but not Hausdorff.

Ex Uncountable set with discrete topology is locally Euclidean ($\dim = 0$) but has no countable basis.

Def 1) M a topological manifold: a *chart* on M is a pair (U, x) with

- $U \subset M$ open
- $x : U \to \mathbb{A}^n$ homeomorphism onto $x(U)$, with $x(U) \subset \mathbb{A}^n$ open

2) Charts (U, x) and (V, y) are *C^∞-related* if



$y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$

and its inverse

$x \circ y^{-1} : y(U \cap V) \to x(U \cap V)$

are both C^∞ maps.

Ex Let $S^2 \subset \mathbb{A}^3$ be the set $\{(a,b,c) : a^2 + b^2 + c^2 = 1\}$

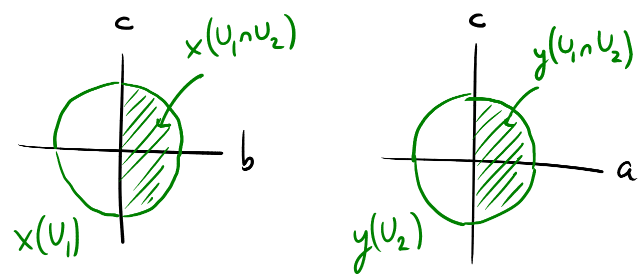
$U_1 = \{a > 0\} \subset S^2$

$U_2 = \{b > 0\} \subset S^2$

$U_1 \cap U_2 = \{a > 0, b > 0\}$

$(U_1 ; \overset{x}{\overset{\shortparallel}{(b,c)}})$ and $(U_2 ; \overset{y}{\overset{\shortparallel}{(a,c)}})$ are charts.

Each has image a disc in $\mathbb{A}^2$.



Overlap map: $(a,c) \mapsto (b = \sqrt{1 - a^2 - c^2}, c)$

Inverse map: $(b,c) \mapsto (a = \sqrt{1 - b^2 - c^2}, c)$

both smooth

so these charts are C^∞-related.

Rk

- These 2 charts don't cover S^2 — would need 6 similar charts to do this.
- In HW, make a covering of S^2 by 2 C^∞-related charts.
- $\nexists$ covering of S^2 by a single chart. Indeed, if (S^2, x) is a chart then $x(S^2)$ is open in $\mathbb{A}^2$ and also compact $\#$
- Can similarly cover $S^n = \{(x^1)^2 + \cdots + (x^{n+1})^2 = 1\} \subset \mathbb{A}^{n+1}$ by $2(n+1)$ charts.

Def M topological manifold: a <u>smooth atlas</u> on M is a collection $\mathcal{C} = \{(U_\alpha, x_\alpha)\}_{\alpha \in \mathcal{A}}$ of charts such that

1) $\bigcup_{\alpha \in \mathcal{A}} U_\alpha = M$

2) If $\alpha_1, \alpha_2 \in \mathcal{A}$ then $(U_{\alpha_1}, x_{\alpha_1})$ and $(U_{\alpha_2}, x_{\alpha_2})$ are C^∞-related

3) $\mathcal{C}$ is maximal wrt 1), 2)

Prop Any $\mathcal{C}$ obeying 1), 2) can be uniquely extended to a smooth atlas.

Pf The extended atlas consists of all (U, x) which are C^∞-related to everything in $\mathcal{C}$.
(Check this atlas still satisfies 2)!) ∎

Ex
1) $\{(\mathbb{A}^1, x)\}$ completes to a smooth atlas on $\mathbb{A}^1$.
2) $\{(\mathbb{A}^1, x^3)\}$ " " a <u>different</u> smooth atlas $\mathcal{C}'$ on $\mathbb{A}^1$, with $x \notin \mathcal{C}'$.
3) $\{(\mathbb{A}^n, x)\}$ completes to a smooth atlas on $\mathbb{A}^n$. ("standard")
4) $\{$the 6 charts discussed above on $S^2\}$ completes to a smooth atlas on S^2.

Def A <u>smooth manifold</u> is a pair $(M, \mathcal{C})$: M a topological manifold, $\mathcal{C}$ a smooth atlas on M.

Usually don't write $\mathcal{C}$ explicitly, use "smooth chart on M" to mean "chart in $\mathcal{C}$"

Def M, N smooth manifolds, $f: M \to N$:

1) $p \in M$: f is <u>smooth at p</u> if, $\forall$ smooth charts (U, x) on M, (V, y) on N with $p \in U$

$y \circ f \circ x^{-1}: x(U \cap f^{-1}(V)) \to y(V)$ is smooth at $x(p)$.

2) f is <u>smooth</u> if $\forall p \in M$ f is smooth at p.

3) f is <u>diffeomorphism</u> if f is bijective and both f and f^{-1} are smooth.

Fortunately we don't really have to check <u>all</u> charts:

Prop f is smooth at $p \iff \exists$ smooth charts (U, x) on M, (V, y) on N with $p \in U$, $y \circ f \circ x^{-1}$ smooth at $x(p)$.

Pf If (U', x') is another chart on M and (V', y') another on N,
then $y' \circ f \circ x'^{-1} = (y' \circ y^{-1}) \circ (y \circ f \circ x^{-1}) \circ (x \circ x'^{-1})$ composition of smooth functions. ∎

Ex The antipodal map $S^2 \to S^2$ is smooth: it takes $(a,b) \to (-a,-b)$

 $\begin{bmatrix}\text{in } c>0\\ \text{chart}\end{bmatrix}$ $\begin{bmatrix}\text{in } c<0\\ \text{chart}\end{bmatrix}$ 

Rk A given topological manifold may admit many non-diffeomorphic smooth atlases!

e.g. $M = \mathbb{A}^4$ is like this (but <u>not</u> $\mathbb{A}^n$ for any $n \neq 4$!)

Or, it may admit none at all.

Rk Guillemin–Pollack def^n of manifold involves taking M to be $\subset \mathbb{A}^N$ for some N. We'll prove later that theirs is <u>equivalent</u> to our def.

Ex

1) If M is a smooth mfd and $U \subset M$ open, then U is smooth mfd (HW)
2) $GL_n\mathbb{R} = \{n \times n \text{ matrices } A \text{ with } \det A \neq 0\}$ is open in $\mathbb{R}^{n^2}$, hence is smooth mfd; similarly $GL_n\mathbb{C}$ is open in $\mathbb{C}^{n^2} \simeq \mathbb{R}^{2n^2}$
3) If M, N are smooth mfds then $M \times N$ is a smooth mfd. (HW)