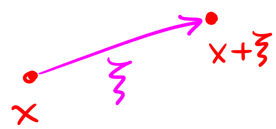
## Vector spaces and affine spaces

This room is *not* a vector space: what does it mean to "add" two points?
The precise structure it has is well captured by the notion of *affine space*.

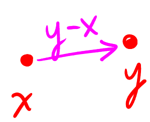
**Def** $V$ a vector space (over $\mathbb{R}$): an *affine space $A$ over $V$* is a set with a free transitive action of $V$.

Think of $x \in A$ as *points*, $\xi \in V$ as *arrows*.
Write the action as $+ : A \times V \longrightarrow A$ $\qquad x \in A,\ \xi \in V \rightsquigarrow x + \xi \in A$




Operations:

$$+ : V \times V \longrightarrow V \qquad + : A \times V \longrightarrow A$$
$$- : V \times V \longrightarrow V \qquad - : A \times A \longrightarrow V$$
$$\cdot : \mathbb{R} \times V \longrightarrow V$$



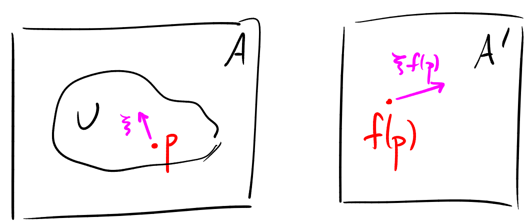

Standard example:

$$V = \mathbb{R}^n = \{(\xi^1, .., \xi^n) : \xi^i \in \mathbb{R}\}$$
$$A = \mathbb{A}^n = \{(x^1, .., x^n) : x^i \in \mathbb{R}\}$$

$+ : A \times V \longrightarrow A$ is componentwise addition

### Derivatives

Say $A$ affine space over $V$, $A'$ affine space over $V'$,
$U \subset A$ open, $f : U \to A'$, $p \in U$, $\xi \in V$.




**Def** (Directional derivative) $\displaystyle \xi f(p) = \lim_{t \to 0} \frac{f(p + t\xi) - f(p)}{t} \in V'$ (if this limit $\exists$)

**Thm/Def** If $\xi f(p)$ exists $\forall p \in U, \xi \in V$ and $\xi f : U \to V'$ is continuous $\forall \xi \in V$ then $\xi \mapsto \xi f(p)$ is a linear function of $\xi$.

It is denoted $df_p$.

**Pf** On choosing a basis, this becomes a standard statement from multivariate calculus. ∎

So, $df_p : V \to V'$, ie $df_p \in \mathrm{Hom}(V, V')$

or, $df : U \to \mathrm{Hom}(V, V')$

**Def** $f$ is smooth in $U$ if, $\forall s \in \mathbb{Z}_{\geq 0}$ and $\xi_1, \ldots, \xi_s \in V$, $\xi_1 \cdots \xi_s f \in V'$ exists.

**Thm** If $f$ is smooth on $U$, and $\xi_1, \xi_2 \in V$, then $\xi_1 \xi_2 f = \xi_2 \xi_1 f$.

**Pf** On choosing a basis, this follows from the equality of mixed partials. ∎

**Thm** If $f : A \to A'$ and $g : A' \to A''$ smooth then $g \circ f$ is smooth and $[d(g \circ f)]_p = (dg)_{f(p)} \circ (df)_p$

**Pf** On choosing a basis, this is the multivariable chain rule. ∎