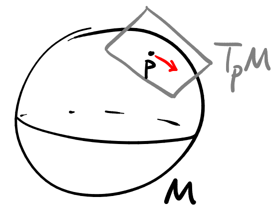
Tangent spaces

$f: M \to N$ should have a corresponding df which maps "vectors on M based at p" to "vectors on N based at $f(p)$"

$$df: T_pM \to T_{f(p)}N$$

"tangent space"



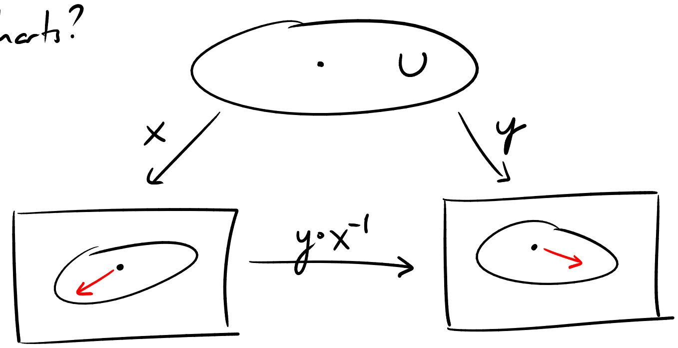
How to formulate this?

Idea: given a chart (U, x) on M with $p \in U$, should be able to identify T_pM with the vector space $\mathbb{R}^n$.

But how do we compare different charts?

Use the map $d(y \circ x^{-1})$.

Precisely:



Def M a smooth mfd with atlas $\mathcal{C} = \{(U_\alpha, x_\alpha)\}_{\alpha \in A}$, $p \in M$: let $m = \dim_M(p)$

$T_pM = \{(\alpha, \xi) : \alpha \in A \text{ with } p \in U_\alpha, \xi \in \mathbb{R}^m\}/\sim$ where $(\beta, \xi) \sim (\alpha, d(x_\alpha \circ x_\beta^{-1})\xi)$

Prop T_pM is a vector space, with a canonical isomorphism $j_\alpha : T_pM \xrightarrow{\sim} \mathbb{R}^m$ for each $\alpha \in A$ with $p \in U_\alpha$, and $j_\alpha \circ j_\beta^{-1} = d(x_\alpha \circ x_\beta^{-1})_{x_\beta(p)}$.

Pf Vector space structure: $(\alpha, \xi) + (\alpha, \xi') = (\alpha, \xi + \xi')$ and $\lambda \cdot (\alpha, \xi) = (\alpha, \lambda \xi)$

Define $j_\alpha(\alpha, \xi) = \xi$. Check it's $\simeq$.

Then $j_\alpha \circ j_\beta^{-1}(\xi) = j_\alpha(\beta, \xi) = j_\alpha(\alpha, d(x_\alpha \circ x_\beta^{-1})\xi) = d(x_\alpha \circ x_\beta^{-1})\xi$. $\blacksquare$

Practically speaking, this means: given a chart (U_α, x_α) we get a basis for T_pM.

Call the basis vectors $e_{\alpha i}$, $i=1,\dots,m$. The bases are related by $e_{\alpha i} = \dfrac{\partial x^j_\beta}{\partial x^i_\alpha} e_{\beta j}$

(implicit summation over j) (More precisely, "$\dfrac{\partial x^j_\beta}{\partial x^i_\alpha}$" means $\dfrac{\partial}{\partial x^i}\left[x_\beta \circ x_\alpha^{-1}\right]^j$). component of the map $\mathbb{A}^m \to \mathbb{A}^m$

coordinates on $\mathbb{A}^m$

Def/Prop If $f: M \to N$ smooth, we define $df_p: T_pM \to T_{f(p)}N$ by

$$df_p = j_{\alpha'}^{-1} \circ d\left[x_{\alpha'} \circ f \circ x_\alpha^{-1}\right] \circ j_\alpha$$

for any α s.t. $p \in U_\alpha$ and α' s.t. $f(p) \in V_{\alpha'}$

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle x_\alpha}\downarrow & & \downarrow{\scriptstyle x_{\alpha'}} \\ \mathbb{A}^m & \longrightarrow & \mathbb{A}^n \end{array} \qquad \begin{array}{ccc} T_pM & \xrightarrow{df} & T_{f(p)}N \\ {\scriptstyle j_\alpha}\downarrow & & \downarrow{\scriptstyle j_{\alpha'}} \\ \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \end{array}$$

This is well defined (doesn't depend on which α, α' we choose).

Pf Changing α to β and α' to β',

$$j_{\beta'}^{-1} \circ d\left[x_{\beta'} \circ f \circ x_\beta^{-1}\right] \circ j_\beta = j_{\alpha'}^{-1} \circ \underbrace{j_{\alpha'} \circ j_{\beta'}^{-1}}_{d(x_{\alpha'} \circ x_{\beta'}^{-1})} \circ d\left[x_{\beta'} \circ f \circ x_\beta^{-1}\right] \circ \underbrace{j_\beta \circ j_\alpha^{-1}}_{d(x_\beta \circ x_\alpha^{-1})} \circ j_\alpha$$

$$= j_{\alpha'}^{-1} \circ d\left[x_{\alpha'} \circ f \circ x_\alpha^{-1}\right] \circ j_\alpha \qquad \blacksquare$$

Def If $f: M \to \mathbb{R}$ smooth then get

$df_p: T_pM \to T_{f(p)}\mathbb{R} \simeq \mathbb{R}$ [since $\mathbb{R}$ has a canonical chart]

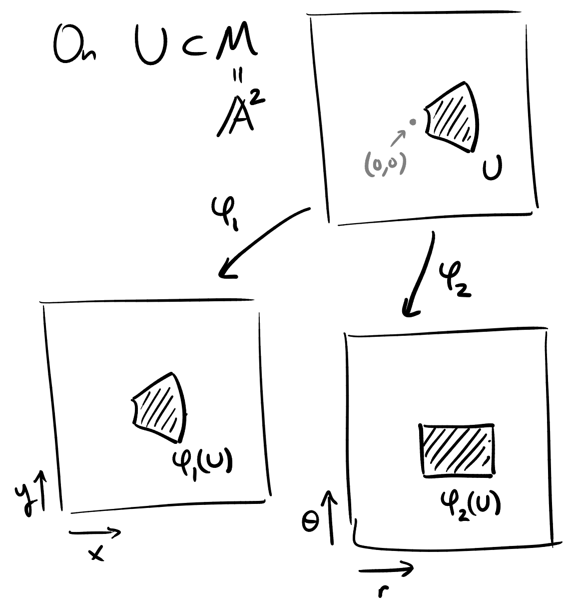
For $\xi \in T_pM$ we write ξf for $df_p(\xi)$ (directional derivative).

Prop If $\xi = \xi^i_\alpha e_{\alpha i}$ then $\xi f = \xi^i_\alpha \dfrac{\partial f}{\partial x^i_\alpha}$. (more precise: $\dfrac{\partial f}{\partial x^i_\alpha}$ means $\dfrac{\partial}{\partial x^i} f \circ x_\alpha^{-1}$)

Pf Exercise.

Rk Thus the basis vectors for T_pM which we called $e_{\alpha i}$ could equally well have been called $\dfrac{\partial}{\partial x^i_\alpha}$. This is a popular notation.

Ex On $U \subset M = \mathbb{A}^2$



two charts: $(U, \underbrace{(x,y)}_{=\varphi_1})$ $\quad (U, \underbrace{(r,\theta)}_{=\varphi_2})$

overlap $\varphi_1 \circ \varphi_2^{-1} : (r,\theta) \mapsto (x,y) = (r\cos\theta, r\sin\theta)$ ie, $x = r\cos\theta$, $y = r\sin\theta$

T_pM has 2 bases: $\left\{ \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y} \right\}$ and $\left\{ \dfrac{\partial}{\partial r}, \dfrac{\partial}{\partial \theta} \right\}$

related by eg

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r} \cdot \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \cdot \frac{\partial}{\partial y} = \cos\theta \cdot \frac{\partial}{\partial x} + \sin\theta \cdot \frac{\partial}{\partial y}$$

Rk There are alternative approaches: e.g.

1) define T_pM as a space of <u>derivations</u> acting on C^∞ functions on M
2) define T_pM as a space of <u>equivalence classes of curves</u> in M passing thru p

All are <u>equivalent</u>.