Some preparation for Inverse Function Theorem:

Def 1) V vector space: a norm $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ is a function s.t.

- $\|\xi\| = 0 \iff \xi = 0$,
- $\|\lambda\xi\| = \lambda\|\xi\|$, $\quad \lambda \in \mathbb{R},\ \xi \in V$
- $\|\xi_1 + \xi_2\| \leq \|\xi_1\| + \|\xi_2\|$.

2) If $T : V \to W$ linear and V, W have norms then

$$\|T\| = \inf\{C \in \mathbb{R}_{\geq 0} : \|T\xi\| \leq C\|\xi\| \ \forall \xi \in V\} = \sup\left\{\frac{\|T\xi\|}{\|\xi\|} : \xi \in V, \xi \neq 0\right\}$$

Rk 1) A normed vector space becomes a metric space by $d(\xi, \xi') = \|\xi - \xi'\|$

2) If V finite-dimensional $\|T\| < \infty$ (use compactness of sphere)

3) On $\mathbb{R}^n$ can take e.g. $\|(\xi^1, \dots, \xi^n)\| = \sup |\xi^j|$ or $\left(\sum_{i=1}^n |\xi^i|^p\right)^{\frac{1}{p}}$, $\ 1 \leq p$

Prop V finite-dim^l: given $f : [a,b] \to V$ cts, we may define $\int_a^b f\, dt \in V$, obeys

1) $\int_a^b \frac{df}{dt}\, dt = f(b) - f(a)$

2) $\left\|\int_a^b f\, dt\right\| \leq \int_a^b \|f\|\, dt$ $\qquad$ (for any norm on V)

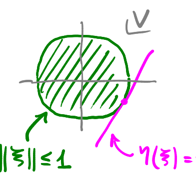
Pf To define it, "work in components" — i.e. for any $\eta \in V^*$ we can define $I(\eta) = \int_a^b \eta(f)\, dt$, then check $I : V^* \to \mathbb{R}$ is linear ie $I \in V^{**}$, then use $V^{**} \simeq V$ to get $I \in V$, define $\int_a^b f\, dt = I$.

↑ since V is finite-dim^l

Then 1) is easy.

For 2), pick $\eta \in V^*$ with $\|\eta\| = 1$ and $I(\eta) = \|I\|$. (This exists for any $\|\cdot\|$ by "Supporting Hyperplane Thm" but it's easy to get directly if we use standard norm on $\mathbb{R}^n$ wrt some basis)

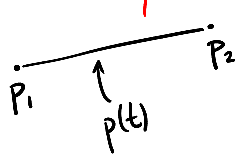
Then, $\|I\| = I(\eta) = \int_a^b \eta(f)\, dt \leq \int_a^b |\eta(f)|\, dt \leq \int_a^b \|f\|\, dt$. ∎



Lemma $f : U \to A'$, A, A' affine spaces, $U \subset A$ a ball, V, V' normed:

If $\|df_p\| < c$ $\forall p \in U$, then $\|f(p_1) - f(p_2)\| \leq c\|p_1 - p_2\|$.

Pf



Set $g(t) = f(p(t))$. Then

$p(t) = p_1 + t(p_2 - p_1)$

$$f(p_2) - f(p_1) = g(1) - g(0) = \int_0^1 g'(t)\, dt = \int_0^1 dg_t\left(\tfrac{\partial}{\partial t}\right) dt$$

$$= \int_0^1 df_{p(t)}\left(dp_t(\partial/\partial t)\right) dt = \int_0^1 df_{p(t)}(p_2 - p_1)\, dt$$

so $\|f(p_2) - f(p_1)\| \leq \int_0^1 \|df_{p(t)}(p_2 - p_1)\|\, dt \leq \int_0^1 c \cdot \|p_2 - p_1\|\, dt = c\|p_2 - p_1\|$ ∎

Lemma If X is a complete metric space, $0<c<1$, and $\phi : X \longrightarrow X$ has $d(\phi(p),\phi(q)) < c\, d(p,q)$ then $\exists !\, p \in X$ s.t. $\phi(p) = p$.

Pf Uniqueness easy.

For existence: fix $x_0 \in X$, set inductively $x_{n+1} = \phi(x_n)$

Then by induction $d(x_{n+1}, x_n) \leq c^n \cdot d(x_1, x_0)$

and for $n<m$, $d(x_n, x_m) \leq \sum_{i=n+1}^{m} d(x_i, x_{i+1}) \leq (c^n + c^{n+1} + \cdots + c^{m-1})\, d(x_1,x_0) \leq \frac{c^n}{1-c} d(x_1,x_0)$

Thus $\{x_n\}$ is Cauchy, $x_n \longrightarrow x$ for some x. $\phi(x) \underset{\uparrow}{=} \lim_{n\to\infty} \phi(x_n) = \lim_{n\to\infty} x_{n+1} = x.$

ϕ cts

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