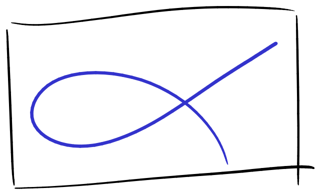
Warning: even though immersions are $\underline{\text{locally}}$ nice, like $\mathbb{A}^m \hookrightarrow \mathbb{A}^n$

they may not be $\underline{\text{globally}}$ nice!

$\underline{\text{Def}}$ $f: M \to N$ is a (smooth) $\underline{\text{embedding}}$ if

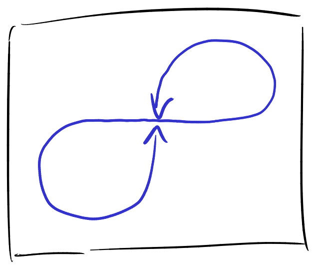
1) f is an immersion at every $p \in M$
2) f is injective
3) f is a homeomorphism onto $f(M) \subset N$ with the subspace topology

$\underline{\text{Ex}}$ a) $\mathbb{A}^1 \to \mathbb{A}^2$



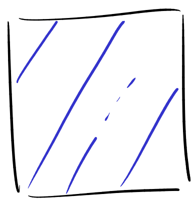
immersion but not 1-1

b) $\mathbb{A}^1 \hookrightarrow \mathbb{A}^2$



immersion, 1-1, but not embedding
(f^{-1} not cts in subspace topology)

c) $\mathbb{A}^1 \hookrightarrow \underbrace{\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}}_{\text{2-torus}}$ — (this is a manifold!)



$x \mapsto (x, \alpha x)$ for α irrational

immersion, 1-1, but not embedding
(f^{-1} not cts in subspace topology)

$\underline{\text{Def}}$ $f: X \to Y$ is $\underline{\text{proper}}$ if, $\forall K \subset Y$ compact, $f^{-1}(K) \subset X$ is compact.

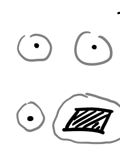
$\underline{\text{Ex}}$ b), c) above are not proper.

$\underline{\text{Ex}}$ If X is compact then $\underline{\text{any}}$ cts $f: X \to Y$ is proper.

Since compactness came up here, a little detour:

$\underline{\text{Lemma}}$ If M is a mfd, then M is metrizable.

$\underline{\text{Pf}}$ Urysohn Metrization Thm: any second-countable T_3-space is metrizable. [T_3: can separate points from points $\underline{\text{and}}$ can separate points from closed sets — easy to see it's true for manifolds



] ∎

$\underline{\text{Cor}}$ If M is a mfd, then $K \subset M$ compact $\iff$ $K \subset M$ sequentially compact (every seq. in K has a convergent subseq.)

$\underline{\text{Pf}}$ This is true for any metric space (see e.g. Baby Rudin) ∎

Prop If $f: M \to N$ is a proper injective immersion, then f is an embedding.

Pf Want to show f is homeomorphism onto $f(M)$. Just need to see f^{-1} is cts, ie, $U \subset M$ open $\Rightarrow$ $f(U) \subset f(M)$ open.

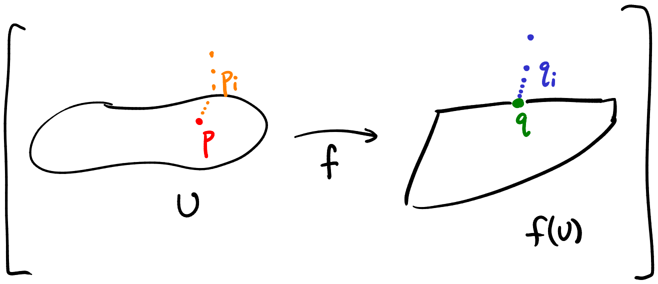
Suppose $f(U)$ not open, then $\exists$ points $q_i \notin f(U)$, $q_i \to q$, $q \in f(U)$.

The set $\{q_i\} \cup \{q\}$ is compact (because sequentially compact). So $f^{-1}(\{q_i\} \cup \{q\})$ is compact by properness of f.

Say $p_i = f^{-1}(q_i)$, $p = f^{-1}(q)$. After passing to a subsequence, by compactness, we have $p_i \to \tilde{p}$ for some $\tilde{p}$.

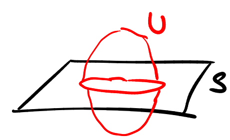
Here $\underset{q_i}{\underset{\parallel}{f(p_i)}} \to f(\tilde{p})$, so $f(\tilde{p}) = q$, so $\tilde{p} = p$. But U open, so this implies $p_i \in U$ for large enough i. ※

Thus $f(U)$ is open ∎



Def $S \subset N$ is a (smooth) submanifold of N if, for every $p \in S$, there exists a chart (U, ϕ) of N with $p \in U$, and some $m \in \mathbb{Z}_{\geq 0}$, s.t.

$$\phi(U \cap S) = \{(x^{m+1} = \cdots = x^n = 0)\} \subset \phi(U).$$



(Slogan: a smooth submfd is cut out by the vanishing of a collection of $n-m$ "independent" functions)

Prop If $S \subset N$ is a smooth submfd of N, S has a natural structure of smooth manifold, and the inclusion $\iota: S \hookrightarrow N$ is an embedding.

Pf For $p \in S$ let (U, ϕ) be chart in def of submfd. Then if $\pi: \mathbb{A}^n \to \mathbb{A}^m$ is the projection, $(U \cap S, \pi \circ \phi)$ gives a chart on S containing p. They're all C^∞-related; complete them to atlas.

To check ι is embedding, use charts $\underset{x}{\underset{\parallel}{(U, \phi)}}$ and $\underset{y}{\underset{\parallel}{(U \cap S, \pi \circ \phi)}}$. Then $y \circ \iota \circ x^{-1}$ is inclusion $\mathbb{A}^m \hookrightarrow \mathbb{A}^n$. ∎

Prop If $f: M \to N$ is an embedding then $f(M)$ is a smooth submanifold of N.

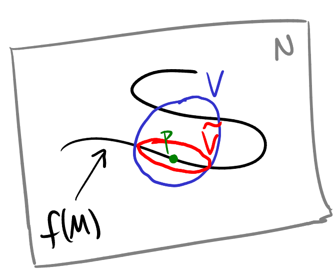
Pf For $p \in M$, pick charts (U, x) and (V, y) s.t. $p \in U$, $f(U) \subset V$, and $y \circ f \circ x^{-1}$ is inclusion $\mathbb{A}^m \hookrightarrow \mathbb{A}^n$, restricted to $x(U)$ (such charts $\exists$ by Thm on local behavior of immersions).

This is almost already what we want — only trouble is that maybe $f(M) \cap V$ is bigger than $f(U) \cap V$.

Since f is embedding, $f(U)$ is open in $f(M)$, i.e. $\exists \tilde{V} \subset N$ open such that $f(U) = \tilde{V} \cap f(M)$.

Then $(\tilde{V}, y)$ is the desired chart in N, with $f(p) \in \tilde{V}$, s.t.

$$y(\tilde{V} \cap f(M)) = y(f(U)) = \{(x^{m+1} = \cdots = x^n = 0)\} \subset y(\tilde{V})$$



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