# Fiber bundles

**Def** $M, F$ smooth manifolds:

1) a fiber bundle over $M$ with fiber $F$ is a surjective submersion $\pi: E \to M$ such that $\forall p \in M$, $\exists$ nbhd $U$ of $p$ and a diffeo $\varphi: \pi^{-1}(U) \to U \times F$ such that $\pi(\varphi^{-1}(u,f)) = u$ ("local trivialization").

2) a map of fiber bundles is $\phi: E \to E'$ s.t. $\pi' \circ \phi = \pi$

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

**Ex** 1) The Hopf map $\pi: S^3 \to S^2$ (from HW) is a fiber bundle with fiber $S^1$!

(Exercise: construct the local trivializations $\varphi$.)

2) The tangent bundle $\pi: TM \to M$ is a fiber bundle with fiber $\mathbb{R}^{\dim M}$.

(local trivializations: for any $p \in M$, pick a chart $(U_\alpha, x_\alpha)$ around $p$, then define $\varphi(p, [(\alpha, \xi)]) = (p, \xi)$)

3) $\pi: M \times F \to M$ is a fiber bundle ("trivial bundle")

**Notation** If $\pi: E \to M$ fiber bundle, write $E_p$ for $\pi^{-1}(p)$ ("fiber at $p$"), and write $E|_U$ for the fiber bundle over $U \subset M$, given by $\pi|_U : \pi^{-1}(U) \to U$.

**Def** $\pi: E \to M$ fiber bundle: a smooth section is $s: M \to E$ smooth such that $\pi \circ s = 1$. Let $\Gamma(E) = \{\text{smooth sections of } E\}$.

**Rk** Smooth sections always exist locally: given local triv $\varphi$, take any smooth map $f: U \to F$, then $s(u) = \varphi^{-1}(u, f(u))$ is a section of $E|_U$.

But smooth sections need not exist globally.

**Ex** Realize $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Define $\pi: S^1 \to S^1$ by $\pi(z) = z^2$.

This is a fiber bundle with fiber $F = $ (2 points). But it doesn't have a global smooth section (if it did, the total space $E$ would be disconnected)

**Def** $M$ smooth manifold:

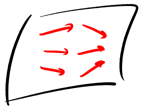
1) a smooth vector bundle $E$ over $M$ is a fiber bundle with fiber $\mathbb{R}^k$, such that each $E_p$ is a vector space, and the local trivializations $\varphi: \pi^{-1}(U) \to U \times \mathbb{R}^k$ are linear.

2) a map of vector bundles over $M$ is a map of fiber bundles $\phi: E \to E'$ where $E, E'$ are vector bundles and $\phi: E_p \to E'_p$ is linear $\forall p \in M$.

**Ex** Example 2) above is vector bundle, as is 3) if $F = \mathbb{R}^k$.

**Rk** If $E$ smooth v.b. then $\Gamma(E)$ is a vector space

**Def** A <u>smooth vector field</u> on $M$ is a smooth section of $TM$.



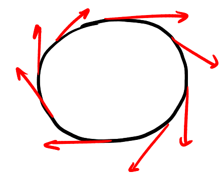
**Rk** 1) Every vector bundle admits a section, the <u>zero section</u>.

2) $\forall p \in M$, $\exists U$ nbhd of $p$ and sections $s_1, \dots, s_k$ of $E|_U$ s.t. $\forall p' \in U$ $\{s_1(p'), \dots, s_k(p')\}$ are a basis of $E_{p'}$. (from the local triv)

3) If such sections $\exists$ over the whole $M$, they give a map of vector bundles

$$\phi : E \xrightarrow{\sim} M \times \mathbb{R}^k .$$

4) The section $\frac{\partial}{\partial \theta}$ of $TS^1$ gives $TS^1 \simeq S^1 \times \mathbb{R}$ (HW)



**Def/Prop** $E$ vector bundle over $M$: $E^*$ is the vector bundle over $M$ defined as follows.

$$E^* = \bigsqcup_{p \in M} E_p^* \qquad \text{with obvious } \pi : E^* \to M$$

Loc triv: for each loc. triv. $\phi : E|_U \to U \times \mathbb{R}^k$ $\qquad \underset{\in E_p}{\phi(e)} = (p, \phi_p(e))$

define loc. triv. $\tilde{\phi} : E^*|_U \to U \times \mathbb{R}^k$ $\qquad \underset{\in E_p^*}{\tilde{\phi}(\eta)} = (p, \tilde{\phi}_p(\eta))$

by $\tilde{\phi}_p(\eta) \cdot \phi_p(e) = \eta \cdot e$ (dot product in $\mathbb{R}^k$; duality pairing $E_p^* \times E_p \to \mathbb{R}$)

Topology, smooth manifold structure on $E^*$: $\exists!$ structure s.t. all $\tilde{\phi}$ are diffeos.

**Pf** See next exercise set.

**Prop** Given $s \in \Gamma(E)$ and $\eta \in \Gamma(E^*)$, the function $\eta \cdot s : M \to \mathbb{R}$, $p \mapsto \eta(p) \cdot s(p)$ is smooth.

**Pf** Exercise.

**Def** $M$ smooth mfd: $(TM)^*$ is <u>cotangent bundle to $M$</u>, also called $T^*M$.

**Rk** Given smooth $f : M \to \mathbb{R}$

we obtain $\quad df_p : T_pM \to T_{f(p)}\mathbb{R} \simeq \mathbb{R}$

i.e. $\quad df_p \in (T_pM)^*$

In fact $df$ is a <u>smooth section</u> of $T^*M$. ("$df_p$ varies smoothly with $p$")

For any coordinate chart $(U, x)$ have sections $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \in \Gamma(TM|_U)$

also have sections $dx^1, \dots, dx^n \in \Gamma(T^*M|_U)$

these are <u>dual</u> <u>bases</u>, in the sense that $\frac{\partial}{\partial x^i} \cdot dx^j = \delta_i^j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

(<u>constant function</u> on $U$)