Transversality theorem

Rk So far we considered homotopies $F: M\times[0,1]\to N$

Sometimes convenient to deform in a bigger family — $F: M\times S\to N$

("homotopy parameterized by S")

Basic tool: a way of perturbing a map $f: M\to N$ to become transverse to some $Q\subset N$.

Prop M manifold with boundary, N, S manifolds, $Q\subset N$ submanifold,

$F: S\times M\to N$ with $F, \partial F \pitchfork Q$:

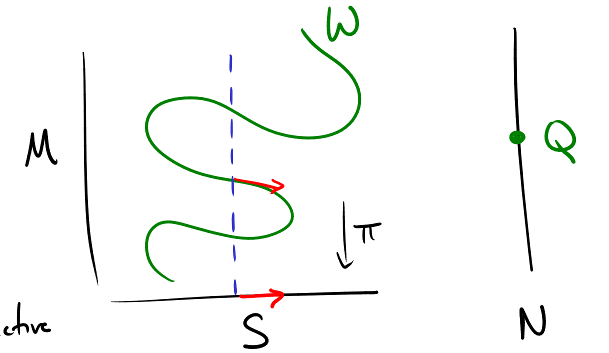
$F_\sigma: M\to N$

$\partial F_\sigma: \partial M\to N$

Then, for almost every $\sigma\in S$, $F_\sigma, \partial F_\sigma \pitchfork Q$.

Pf Let $W = F^{-1}(Q)\subset S\times M$.

$$\begin{tikzcd} & W \arrow[dl, "\pi"'] \arrow[dr] & \\ S & & M \end{tikzcd}$$



Geometric picture in special case $Q=pt$:

looks like $F_\sigma \pitchfork Q \iff d\pi: TW\to TS$ surjective

(relates surjectivity on vertical vectors to surjectivity on vectors tangent to W)

Claim, indeed: $F_\sigma\pitchfork Q, \partial F_\sigma\pitchfork Q \iff \sigma\in S$ regular value for $\pi|_W$.

"Just" linear algebra: $(\sigma,p)\in S\times M$ $F(\sigma,p)=q\in Q$

Take adapted $G: U\to\mathbb{A}^k$ ($U \ni q$) s.t. $U\cap Q = G^{-1}(0)$. $\left(\begin{aligned} k &= \operatorname{codim} Q \\ &= \dim N - \dim Q \end{aligned}\right)$

Then $F_\sigma, \partial F_\sigma \pitchfork Q$ at $(\sigma,p) \iff (\sigma,p)$ is regular point for $G\circ F_\sigma, G\circ\partial F_\sigma$.

$$\begin{tikzcd} T_{(\sigma,p)}W \subset T_\sigma S\oplus T_pM \arrow[r, "dF_{(\sigma,p)}"] \arrow[rr, bend left, "L"] \arrow[d, "\pi_1"] & T_qN \arrow[r, "dG_q"] & \mathbb{R}^k \\ T_\sigma S & & \end{tikzcd}$$

$T_{(\sigma,p)}W = \ker L$, and L surjective (since $F\pitchfork Q$).

Claim: $\pi_1|_{\ker L}$ surjective $\iff L|_{0\oplus T_pM}$ surjective.

($\Rightarrow$) for $\eta\in\mathbb{R}^k$, $\exists\ (\mu,\xi)$ s.t. $L(\mu,\xi)=\eta$.

also $\exists\ (\mu,\xi')\in \ker L$.

then $L(0,\xi-\xi')=\eta$.

($\Leftarrow$) for $\mu\in T_\sigma S$, $\exists\ \xi\in T_pM$ s.t. $L(0,\xi)=L(\mu,0)$

so $L(\mu,-\xi)=0$.

This proves $F_\sigma \pitchfork Q$ for σ regular value of $\pi|_W$.

Similar for ∂F_σ, just use $T_p\partial M$ instead of T_pM, and transversality of ∂F. $\blacksquare$

How to <u>produce</u> this kind of deformation of a given map f?

If $N=\mathbb{A}^n$, it's easy:

Prop M mfd with boundary, $f: M\to\mathbb{A}^n$: then set $S=\mathbb{A}^n$,

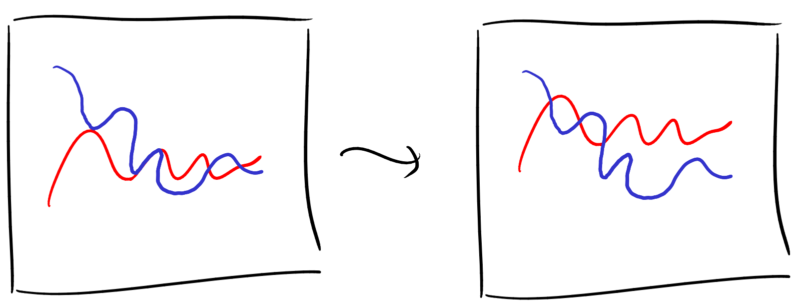
make

$$F: S\times M\to \mathbb{A}^n$$
$$(\sigma,p)\mapsto f(p)+\sigma$$

F is transverse to <u>any</u> $Q\subset\mathbb{A}^n$.

Pf $dF_{(\sigma,p)}$ is already surjective restricted to $T_\sigma S\subset T_{(\sigma,p)}(S\times M)$! $\blacksquare$

Ex Given two curves in $\mathbb{A}^2$, can always make their $\cap$ transverse by <u>translating</u> one of them by a generic distance σ.

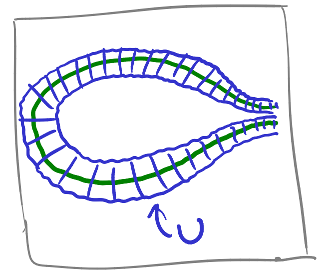


How about more general N?

Can embed $N \subset \mathbb{A}^n$, translate a bit $f \mapsto f + \sigma$ $\sigma \in \mathbb{A}^n$

but then need to <u>project back to</u> N somehow!

Two lemmas we will need (pf later):



Lemma 1 $N \subset \mathbb{A}^k$ submanifold: $\exists$ open nbhd $U \subset \mathbb{A}^k$ of N and submersion $\pi: U \to N$, with $\pi|_N = 1$.

Def For $N \subset \mathbb{A}^n$, $\varepsilon: N \to \mathbb{R}_{>0}$, $B_\varepsilon(N) = \bigcup_{q \in N} B_{\varepsilon(q)}(q) \subset \mathbb{A}^n$
"open ball around N"

Lemma 2 $N \subset \mathbb{A}^k$ submanifold and $U \subset \mathbb{A}^k$ open nbhd of N:
$\exists\, \varepsilon: N \to \mathbb{R}_{>0}$ s.t. $B_\varepsilon(N) \subset U$.

Cor M mfd with boundary, $N \subset \mathbb{A}^k$ submanifold, $f: M \to N$ smooth:
$\exists$ open nbhd $U \subset \mathbb{A}^k$ of N, $\pi: U \to N$, $\varepsilon: N \to \mathbb{R}_{>0}$ s.t.

$$F: S \times M \to N \qquad S = B_1(0) \subset \mathbb{A}^k$$
$$(\sigma, p) \mapsto \pi\big(f(p) + \varepsilon(f(p))\sigma\big)$$

is a submersion and $F_0 = f$.

Pf Just need to check it's submersion. In fact $dF_{(\sigma,p)}: T_\sigma S \to T_{F(\sigma,p)} N$ is already surjective, ($T_\sigma S \subset T_{(\sigma,p)} S \times M$)
since it's $d\pi \circ dg$ $g: S \to U$ and both π, g are submersion
$$\sigma \mapsto f(p) + \varepsilon(f(p))\sigma$$
$$dg: \mathbb{R}^k \to \mathbb{R}^k$$
$$\xi \mapsto \varepsilon(f(p)) \cdot \xi$$
∎

Cor M mfd with boundary, N mfd, $Q \subset N$ submanifold, $f: M \to N$
There exists a homotopy $F: [0,1] \times M \to N$ with $F_0 = f$, such that
$F_1, \partial F_1 \pitchfork Q$.

Pf Use the above $F: S \times M \to N$, pick some $\sigma \in S$ s.t. $F_\sigma, \partial F_\sigma \pitchfork Q$ (using Prop above)
then define $\tilde{F}: [0,1] \times M \to N$
$$(t, p) \mapsto F(t\sigma, p)$$
∎

Now to prove our lemmas.

Need some vector bundle preliminaries.

Notation If $f: X \to Y$ and $g: Y \to Z$ write $f^*g = g \circ f$ ("pullback map")

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f^*g"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Def $f: M \to N$ smooth map, $E \to N$ smooth bundle: f^*E is a vector bundle over M ("pullback bundle") given by $f^*E = \coprod_{p \in M} E_{f(p)}$,

$$\begin{tikzcd} f^*E \arrow[r, "\hat{f}"] \arrow[d, "f^*\pi"'] & E \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

with tautological map $\hat{f}: f^*E \to E$ (not bundle map since $M \neq N$) given by $\hat{f}(\underset{E_{f(p)}}{\xi \in}) = \xi$ — fiberwise $\cong$.

Local trivializations $\tilde{\phi}_\alpha$ induced from local triv $\phi_\alpha: E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^k$ of E:

$$\tilde{\phi}_\alpha: f^*E|_{f^{-1}U_\alpha} \to f^{-1}U_\alpha \times \mathbb{R}^k$$

given by $\tilde{\phi}_\alpha = \hat{f}^*\phi_\alpha$

Rk Alternative description: if E is given by covering $\{U_\alpha\}$ on N and gluings $\psi_{\alpha\beta}: U_\alpha \cap U_\beta \to GL(k,\mathbb{R})$ then f^*E is given by covering $\{f^*U_\alpha\}$ on M and gluings $f^*\psi_{\alpha\beta}$.

Notation If $\iota: M \to N$ inclusion, E vector bundle over N, then write $E|_M$ for ι^*E. ("restriction of vector bundle")

Ex If $f: M \to N$ constant, E vector bundle over N, then f^*E is a trivial bundle over M. (Exercise)

Def E vector bundle over M: a subbundle E' of E is a vector bundle over M such that $E' \subset E$ and the inclusion $\iota: E' \hookrightarrow E$ is a bundle map.

Ex $M \subset N$ submanifold: $TM \subset TN|_M$ is a subbundle.

Def/Prop E vector bundle over M, $E' \subset E$ subbundle: quotient bundle $F = E/E'$ is a bundle such that $F_p = E_p/E'_p$.

Pf Need to construct local trivializations for F.
For $p \in M$ take a nbhd U of p s.t. E and E' are trivial.
This means $\exists$ basis $\{s_1, \ldots, s_k\}$ of sections of $E|_U$
$\{s'_1, \ldots, s'_\ell\}$ " " " $E'|_U$

After reordering, can arrange that $\{s'_1, \ldots, s'_\ell, s_{\ell+1}, \ldots, s_k\}$ are a basis of E_p
hence also a basis of $E_{p'}$ for $p' \in \tilde{U}$, some open $\tilde{U} \subset U$.
Then $\{s_{\ell+1}, \ldots, s_k\}$ are a basis of $E_{p'}/E'_{p'}$ for $p' \in \tilde{U}$.
This gives the desired local trivialization: $F = \coprod_p E_p/E'_p$, $\phi: \quad F|_{\tilde{U}} \to \tilde{U} \times \mathbb{R}^{k-\ell}$

$$\sum_{i=\ell+1}^{k} c_i s_i(p) \mapsto (p, (c_{\ell+1}, \ldots, c_k))$$ ∎

Def $M \subset N$ submanifold: __normal bundle__ $NM = TN|_M / TM$

Rk Have sequence of maps of vector bundles over M:

$$0 \to TM \xrightarrow{\iota} TN|_M \xrightarrow{\pi} NM \to 0$$

This is actually __exact sequence__ of vector bundles: ι injective, π surjective, $\mathrm{Im}(\iota) = \mathrm{Ker}(\pi)$

Then can ask: does the exact sequence split?

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\iota"] & B \arrow[r, "\pi"] & C \arrow[r] \arrow[l, bend right, dashed, "\exists s?"'] & 0 \end{tikzcd}$$

Generally __no__, e.g.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "\mathrm{mod}\, 2"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[l, bend right, dashed, "\nexists s"'] & 0 \end{tikzcd}$$

in category of abelian groups.

But in the category of vector spaces, yes; and even for smooth vector bundles:

Lemma E vector bundle over M, $E' \subset E$ subbundle, $F = E/E'$, $\pi: E \to F$
$\exists$ smooth vector bundle map $F \xrightarrow{s} E$ s.t. $\pi \circ s = 1$.

Pf In next exercise set. (Idea: use partition of unity to "average" local splittings given above)

Rk When $M \subset N = \mathbb{A}^n$ we can split the projection $TN \underset{\pi}{\overset{s}{\rightleftarrows}} TN/TM = NM$ by using dot product in $\mathbb{R}^n$: $s([\xi]) = \xi^\perp \in (T_pM)^\perp \subset T_pN$

This accounts for the name "normal bundle".

Rk For general N, to do this we would need some substitute for the dot product: <u>Riemannian metric</u> on N. [<u>Def</u> A <u>Riemannian metric</u> on M is a symmetric positive def. bilinear $g: T_pM \times T_pM \to \mathbb{R}$, varying smoothly with $p \in M$]

Lemma E vector bundle over M: let $Z \subset E$ be the zero section, $p \in M$, then $Z \pitchfork E_p$ in E.

Pf Pick a local triv (U, φ) for E and chart (U, x) on M. Then we get a chart $(E|_U, y)$ on E by $\underbrace{E|_U \xrightarrow{\varphi} U \times \mathbb{R}^k \xrightarrow{(x,1)} \mathbb{R}^m \times \mathbb{R}^k}_{y}$, and $y(Z \cap E|_U) = \mathbb{R}^m \cap y(E|_U)$, $y(E_p) = \mathbb{R}^k \cap y(E|_U)$ ∎

Cor Let $0_p \in E$ be the zero element of E_p, then $T_{0_p}E \simeq T_pM \oplus E_p$ canonically.

Rk There isn't such a canonical splitting at other points of E.

Rk Often abuse notation by writing $M \subset E$ instead of $Z \subset E$.

Pf of Lemma 1 Idea: identify nbhd of $M \subset \mathbb{A}^k$ with an open nbhd of $M \subset NM$. (zero section)

Define
$$h: NM \longrightarrow \mathbb{A}^k$$
$$\eta \longmapsto \underset{M \subset \mathbb{A}^k}{\pi(\eta)} + \sigma(\eta)$$

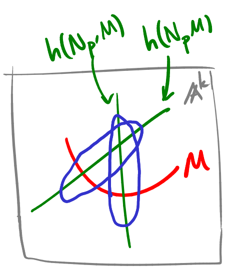
$$\left[\begin{tikzcd} NM \arrow[r, "s"] \arrow[rr, bend right, "\sigma"'] & T\mathbb{A}^k = \mathbb{A}^k \oplus \mathbb{R}^k \arrow[r, "\pi"] & \mathbb{R}^k \end{tikzcd}\right]$$

Then
$$dh_{0_p}: T_{0_p}NM \longrightarrow T\mathbb{A}^k$$
$$\mathbb{R}^k \supset T_pM \oplus N_pM \longrightarrow \mathbb{R}^k$$
$$\xi \oplus \chi \longmapsto \xi + \sigma(\chi)$$

[Why? ① Restrict h to zero section $Z \subset NM$: identifying Z with M, $h|_Z$ is just <u>inclusion</u>, so for $\xi \in T_pM \subset \mathbb{R}^k$, $dh(\xi) = \xi$.
② Restrict h to fiber $N_pM \subset NM$: then h is affine map $\eta \mapsto p + \sigma(\eta)$ so $dh: N_pM \to \mathbb{R}^k$ is just σ]

So h is local diffeo at 0_p.

To finish, need



Lemma $h: M \to N$, local diffeo on submanifold $P \subset M$, <u>injective</u> on P
$\Rightarrow \exists$ open nbhds V of P, U of $h(P)$, s.t. $h: V \xrightarrow{\sim} U$ diffeo.

This is Exercise 14 of 1.10 in G+P but their hint is bogus!

Still, granting this for now, get $U \subset \mathbb{A}^k$ open nbhd of M, with

$$\begin{tikzcd} NM \supset V \arrow[d, "\pi"] \arrow[r, shift right, "h"'] & U \subset \mathbb{A}^k \arrow[l, shift right, "g"'] \\ M & \end{tikzcd}$$

Then $\pi \circ g: U \to M$ is the desired submersion. ∎

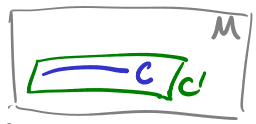
__Pf of Lemma 2__ Naively could just set $\varepsilon(p) = \inf\{\varepsilon : B_\varepsilon(p) \subset U\}$. But this might not be smooth. So:
Choose $\{V_\alpha\}$ open cover of M with $\overline{V}_\alpha$ compact. Then $\forall \alpha$, $\exists\, \varepsilon_\alpha > 0$ s.t. $\forall p \in V_\alpha$, $B_{\varepsilon_\alpha}(p) \subset U$. Let $\{\rho_\alpha\}$ be partition of unity subordinate to $\{V_\alpha\}$.
Set $\varepsilon = \sum_\alpha' \rho_\alpha \varepsilon_\alpha$. Then $\varepsilon(p) \leq \max\limits_{\alpha:\, p \in V_\alpha} \varepsilon_\alpha$, so $B_{\varepsilon(p)}(p) \subset U$ as desired. ∎

Done proving existence of transversal perturbations!

But, need a small __technical improvement__:

__Thm__ M smooth mfd with ∂, N mfd, $Q \subset N$ __closed__ submfd, $C \subset M$ __closed__ subset, $f : M \to N$ s.t. $f, \partial f \pitchfork Q$ at all points of C: then $\exists$ homotopy $F : [0,1] \times M \to N$ with $F_0 = f$, s.t. $F_t = f$ on C, and $F_1, \partial F_1 \pitchfork Q$.

__Pf__ $\exists$ open nbhd U of C s.t. $f, \partial f \pitchfork Q$ at all points of U.



Take C' closed s.t. $C \subset \operatorname{int} C' \subset C' \subset U$.
Then $\{U, M \setminus C'\}$ cover M, take partition of unity $\{\rho_U, \rho_{M\setminus C'}\}$. Set $\tau = \rho_{M \setminus C'}^2$.
Then on C' have $\tau = 0$, $d\tau = 0$ $\quad (d\tau = d(\rho^2) = 2\rho\, d\rho)$
We know $\exists$ perturbation $G : S \times M \to N$ which is __submersion__. $\quad [S = B_1(0) \subset \mathbb{R}^k]$
Now define $F : S \times M \to N$

$$(\sigma, p) \mapsto \begin{cases} G(\tau(p)\sigma, p) & p \in M \setminus C' \\ f(p) & p \in C' \end{cases}$$

Claim: $F, \partial F \pitchfork Q$.

To verify this, note $F = G \circ m$ where $m : S \times M \to S \times M$, $(\sigma, p) \mapsto (\tau(p)\sigma, p)$

For $\tau(p) \neq 0$, m is submersion; thus F is also submersion on $S \times \tau^{-1}(\mathbb{R}_{>0})$.
What about the points where $\tau = 0$?

Here $$dm_{(\sigma,p)}(\underset{T_\sigma S}{\underset{\pitchfork}{\dot\sigma}}, \underset{T_p M}{\underset{\pitchfork}{\dot p}}) = (d\tau_p(\dot p)\sigma + \tau(p)\dot\sigma, \dot p) = (0, \dot p)$$

so $$dF_{(\sigma,p)}(\dot\sigma, \dot p) = dG_{(0,p)}(0, \dot p) = df_p(\dot p)$$

Thus, since $f \pitchfork Q$ at p, $F \pitchfork Q$ at (σ, p). Similar for ∂F. ∎

Cor If M mfd with ∂, N mfd, $Q \subset N$ closed submfd, $\partial f \pitchfork Q$

$\implies \exists\, g: M \to N$, $g \pitchfork Q$, g homotopic to f, $\partial f = \partial g$.