Mod 2 intersection

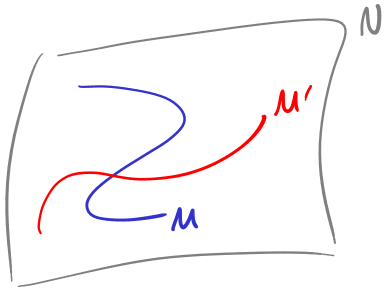
N manifold

2 submanifolds $M, M' \subset N$

$\dim M + \dim M' = \dim N$

at least one of M, M' compact

$\Rightarrow$ "expect" M, M' intersect in a finite # of pts.

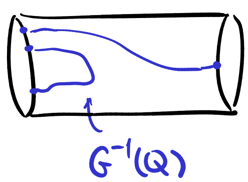


To achieve this, need to perturb to make them transverse.

Def/Prop M, N smooth mfds, M compact, $f: M \to N$ smooth, $Q \subset N$ closed submfd, $\dim M + \dim Q = \dim N$:
$I_2(f, Q) = \#\{g^{-1}(Q)\} \bmod 2$, for some g smoothly homotopic to f with $g \pitchfork Q$.

Pf $g^{-1}(Q)$ is 0-dim closed submfd of $Q \Rightarrow \#\{g^{-1}(Q)\}$ is well defined.
Need to check that if g, g' both homotopic to f, $g \pitchfork Q$, $g' \pitchfork Q$
then $\#\{g^{-1}(Q)\} = \#\{g'^{-1}(Q)\} \bmod 2$.

Let $G: M \times [0,1] \to N$ be homotopy of g and g'. We may assume $G \pitchfork Q$.

Then as for def. of $\deg_2$ earlier,  shows the desired invariance.

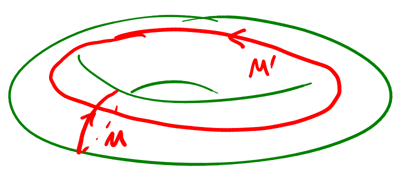
Cor If f, g homotopic then $I_2(f, Q) = I_2(g, Q)$.

Rk When $Q = pt$, $I_2(f, Q) = \deg_2(f)$.

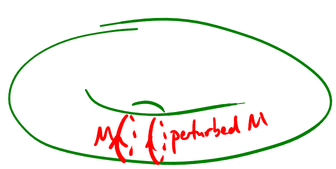
Def If M, M' closed submfds of N, M compact, then define $I_2(M, M') = I_2(\iota, M')$ $\quad \iota: M \hookrightarrow N$ inclusion

If $M \pitchfork M'$ this counts the # of intersections of M and M' mod 2.
If not $M \pitchfork M'$ then have to perturb first to make it transverse.

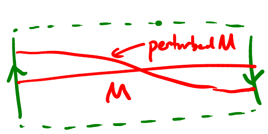
Ex $N = S^1 \times S^1$ $\qquad I_2(M, M') = 1 \bmod 2$
$M = S^1 \times \{0\}$
$M' = \{0\} \times S^1$



$I_2(M, M) = 0 \bmod 2$

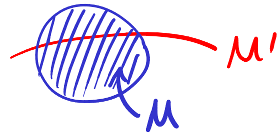


Ex $N =$ Möbius band $\qquad I_2(M, M) = 1 \bmod 2$
$M =$ central curve



Rk For $M \subset N$, $I_2(\partial M, M') = 0 \mod 2$

This follows from:



Thm If $f: \partial M \to N$ can be extended to $\tilde{f}: M \to N$
then $I_2(f, Q) = 0$ $\forall Q \subset N$ closed with $\dim Q + \dim(\partial M) = \dim N$

Pf May assume $\tilde{f} \pitchfork Q$ and $f \pitchfork Q$.
Then $\tilde{f}^{-1}(Q)$ is 1-mfd, $\partial[\tilde{f}^{-1}(Q)] = f^{-1}(Q)$, so $\# f^{-1}(Q) = 0 \mod 2$. ■

Rk It's an exercise in G+P to show that if M, M' both compact
then $I_2(M, M') = I_2(M', M)$.

Ex $N = \mathbb{RP}^2 = \{(x,y,z) \in \mathbb{R}^3\}/\sim$

$M' = \{z = 0\} \subset \mathbb{RP}^2$ linear subspace ($\simeq \mathbb{RP}^1$)

$M = \{f(x,y,z) = 0\}$ f homogeneous degree d, 0 regular value

$I_2(M, M') = ?$

Assume $f(1,0,0) \neq 0$.

Then $M \cap M' = \{(x,y) : f(x,y,0) = 0, y \neq 0\}/\sim \; = \{x : \underbrace{f(x,1,0)}_{P(x) \text{ degree } d \text{ poly.}} = 0\}$

$M \pitchfork M'$ at simple zeroes of $P(x)$. [Exercise]

So, at least if $f(1,0,0) \neq 0$ and all zeroes of $P(x)$ are simple, $M \pitchfork M' = d \mod 2$.

Claim by changing $f \mapsto f \circ A$ for $A \in GL(3)$ $\quad A: \mathbb{A}^3 \to \mathbb{A}^3$
we can always reach this situation. (Needs proof!) But this changes
$\iota: M \to N$ by a homotopy (take path A_t with $A_0 = 1$, $A_1 = A$, and then let $\iota_t = A \circ \iota$)

So $I_2(M, M') = d$ in general.

Ex Show $S^2 \not\cong S^1 \times S^1$, $S^4 \not\cong S^2 \times S^2$.