Orientations

Recall given vector space V, an orientation of V is a connected component of $\det V \setminus \{0\}$.

Rk $\det V^* \simeq (\det V)^*$. Given orientation $\mathcal{O}$ of V, $\{\omega \in \det V^* \mid \omega(\alpha) > 0 \ \forall \alpha \in \mathcal{O}\}$ is an orientation of V^*.
So $\{\text{orientation of } V\} \simeq \{\text{orientations of } V^*\}$ canonically.

Def Given an orientation $\mathcal{O}$ of V, call a basis $\{e_1, \dots, e_n\}$ of V <u>positive</u> (for α) if $e_1 \wedge \cdots \wedge e_n \in \mathcal{O}$.

Prop If $\{e_1, \dots, e_n\}$ and $\{e_1', \dots, e_n'\}$ are both positive, then the matrix A with $e_i' = A_i^j e_j$ has $\det(A) > 0$.

Pf

$$\mathbb{R}^n \overset{e}{\underset{e'}{\rightrightarrows}} V, \qquad A \circlearrowright \mathbb{R}^n, \quad T \circlearrowright V$$

$A = e^{-1}e' = (e')^{-1} T e'$ $\qquad T = e'e^{-1}$

$\det T : \det V \longrightarrow \det V$
takes $e_1 \wedge \cdots \wedge e_n \mapsto e_1' \wedge \cdots \wedge e_n' = \beta\, e_1 \wedge \cdots \wedge e_n$ with $\beta > 0$

Then, $\det A = (\det e')^{-1} (\det T)(\det e')$
takes $f_1 \wedge \cdots \wedge f_n \xmapsto{\det e'} e_1' \wedge \cdots \wedge e_n' \xmapsto{\det T} \beta e_1' \wedge \cdots \wedge e_n' \xmapsto{(\det e')^{-1}} \beta f_1 \wedge \cdots \wedge f_n$
so $\det A : \det \mathbb{R}^n \to \det \mathbb{R}^n$ is mult. by $\beta > 0$. ∎

Def/Prop E vector bundle over M: <u>orientation bundle</u> $\mathrm{or}(E)$ of E is fiber bundle, fiber over $p \in M$ is the set of <u>orientations</u> of E_p.

Pf Given local trivialization (U, φ) of E $\qquad \varphi : E|_U \xrightarrow{\sim} U \times \mathbb{R}^k$
get corresponding loc. triv. $(U, \det \varphi)$ of $\det E$ $\quad \det \varphi : \det E|_U \xrightarrow{\sim} U \times (\det \mathbb{R}^k) \simeq U \times \mathbb{R}$
then divide out by $\mathbb{R}_+$ action on both sides to get $\quad \mathrm{or}\,\varphi : \mathrm{or}\, E|_U \xrightarrow{\sim} U \times \{\pm 1\}$
These are the needed local triv for $\mathrm{or}\, E$ ∎

Def ① An <u>orientation</u> of E is a section of $\mathrm{or}\, E$.
② An <u>orientation</u> of M is an orientation of TM.

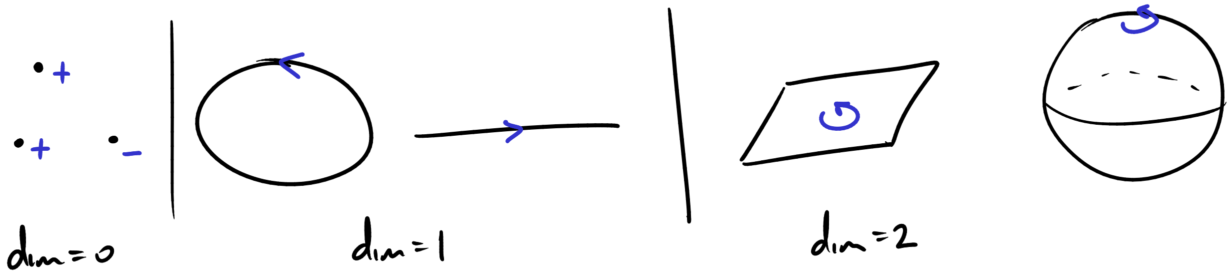
Prop For $\dim(M) > 0$, an orientation of M is equivalent to a covering of M by charts (U_α, x_α) such that for $x \in x_\alpha(U_\alpha) \cap x_\beta(U_\beta)$, $\quad d(x_\alpha \circ x_\beta^{-1})_x : \mathbb{R}^n \to \mathbb{R}^n$ has <u>positive determinant</u>.
(Call these "positively oriented charts")

Pf $(\Leftarrow)$ given p, take a positively oriented chart (U_α, x_α) $p \in U_\alpha$, then take the orientation of T_pM for which $\left\{\frac{\partial}{\partial x_\alpha^1}, \dots, \frac{\partial}{\partial x_\alpha^n}\right\}$ is positively oriented basis.
Check it's indep of choice of α.

$(\Rightarrow)$ if (U, x) is a coord sys, include it in our list of positively oriented charts iff

$\left\{\frac{\partial}{\partial x^1_\alpha}, \dots, \frac{\partial}{\partial x^m_\alpha}\right\}$ is positively oriented basis. (Note this still gives enough positively oriented charts to cover M.) ∎

What oriented manifolds look like:



For higher dimensions a useful fact.

Prop If $0 \to V' \to V \to V'' \to 0$ exact sequence of vector spaces, then $\det V \simeq \det V'' \otimes \det V'$ canonically.

Pf Choose a splitting $s : V'' \to V$.

Then take $\det V'' \times \det V' \longrightarrow \det V$

$$(\beta\ ,\ \alpha) \longmapsto (\Lambda^n s)(\beta) \wedge (\Lambda^m \iota)(\alpha)$$

(key convention: V'' before V' ie quotient before subspace)

Bilinear, so it factors thru a map on $\det V' \otimes \det V''$.

Nontrivial, since $(e_1 \wedge \dots \wedge e_m,\ f_1 \wedge \dots \wedge f_n) \mapsto s(f_1) \wedge \dots \wedge s(f_n) \wedge e_1 \wedge \dots \wedge e_m \neq 0$

To check independence of s: $\tilde{s}(f_1) \wedge \dots \wedge \tilde{s}(f_n) \wedge e_1 \wedge \dots \wedge e_m$

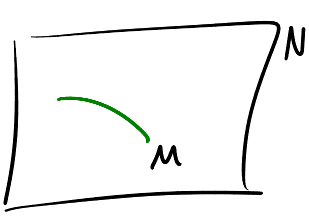
$= s(f_1) \wedge \dots \wedge \tilde{s}(f_n) \wedge e_1 \wedge \dots \wedge e_m$ since $\tilde{s}(f_1) - s(f_1) \in V'$ ie is lin. comb. of e_i

$= \dots = s(f_1) \wedge \dots \wedge s(f_n) \wedge e_1 \wedge \dots \wedge e_m$ ∎

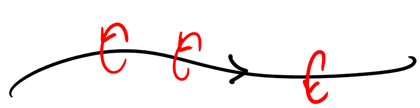
This gives "2 out of 3" rule: any 2 of $\begin{bmatrix} \text{orientation of } V \\ \text{orientation of } V' \\ \text{orientation of } V'' \end{bmatrix}$ determine the 3rd, using:

$\det V' \otimes \det V'' = \det V$
$\det V' = \det V \otimes \det V''^*$
$\det V'' = \det V \otimes \det V'^*$

Ex If $M \subset N$, $\det NM \otimes \det TM \simeq \det (TN|_M)$.



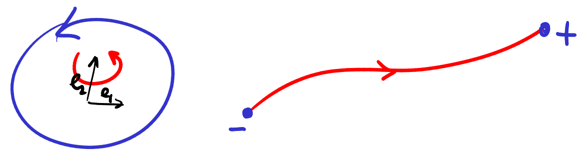
So e.g. fixing an orient^n of 3-mfd N and of a 1-mfd $M \subset N$ induces an orient^n of NM ie "sense of rotation around M"



Orientations on boundary

$0 \to T(\partial M) \to TM|_{\partial M} \to N(\partial M) \to 0$

$N(\partial M)$ has canonical orientation, "outward pointing"
So, an orientation on M induces one on ∂M.



Integration

Say $\omega \in \Omega^n(\mathbb{A}^n)$ with $\operatorname{supp}\omega \subset U$, $\operatorname{supp}\omega$ compact.
Then $\omega = f\, dx^1 \wedge \dots \wedge dx^n$. Equip $\mathbb{A}^n$ with standard orientation, $dx^1 \wedge \dots \wedge dx^n$.

Define $\displaystyle\int_U \omega = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x^1, \dots, x^n)\, dx^1 dx^2 \dots dx^n$

Lemma $\varepsilon = \pm 1$, $\varphi : \underset{\substack{\cap \\ \mathbb{A}^n \\ x^i}}{V} \to \underset{\substack{\cap \\ \mathbb{A}^n \\ y^i}}{U}$ with $\varepsilon \det(d\varphi_x) \geq 0$ $\forall x$: $\displaystyle\int_V \varphi^* \omega = \varepsilon \int_U \omega$.

Pf $\varphi^*(dy^1 \wedge \dots \wedge dy^n) = \det(d\varphi)\, dx^1 \wedge \dots \wedge dx^n$

So $\displaystyle\int_V \varphi^* \omega = \int_V f(y(x)) \det\left(\frac{\partial y^i}{\partial x^j}\right) dx^1 \wedge \dots \wedge dx^n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(y(x)) \det\left(\frac{\partial y^i}{\partial x^j}\right) dx^1 \cdots dx^n$

$\displaystyle = \varepsilon \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(y(x)) \left|\det\left(\frac{\partial y^i}{\partial x^j}\right)\right| dx^1 \cdots dx^n$

$\displaystyle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(y)\, dy^1 \dots dy^n$ by usual change of variable formula (proved using Lebesgue theory) ∎

Def Any chart on a 0-manifold just maps a single point $p \in M$ to $0 \in \mathbb{R}^0$.
We say the chart is positively oriented if p has the $+$ orientation,
negatively oriented if p has the $-$ orientation.

Rk With this understood, orientations on any M are $\simeq$ to rules for dividing charts into +ve and −ve.

Def/Prop For M oriented manifold with boundary and $m = \dim M$, there exists a unique map

$$\int_M : \Omega^m_c(M) \to \mathbb{R}$$

(compact support)

such that

1) $\int_M$ is $\mathbb{R}$-linear,

2) if (U, x) is a chart on M, and $\operatorname{supp}\omega \subset U$, then $\displaystyle\int_M \omega = \varepsilon \int_{x(U)} (x^{-1})^* \omega$ with $\varepsilon = \begin{cases} +1 & \text{if } U \text{ positively or.} \\ -1 & \text{if } U \text{ negatively or.} \end{cases}$

Pf Take (U_i, x_i) loc. finite covering by charts, and a partition of unity $\{\rho_i\}$ rel. to U_i.

1), 2) then determine $\displaystyle\int_M \omega = \sum_i{}' \varepsilon_i \int_{x_i(U_i)} (x_i^{-1})^* (\rho_i \omega)$ $\quad (*)$ $\quad$ [finite sum since ω compactly supported]

$\varepsilon_i = \begin{cases} +1 & \text{if } U_i \text{ positively oriented} \\ -1 & \text{if } U_i \text{ negatively oriented} \end{cases}$

Evidently $(*)$ obeys 1).

Then, suppose (V, y) is some other chart, and $\operatorname{supp} \omega \subset V$.

Then $\displaystyle\int_M \omega = \sum_i \varepsilon_i \int_{x_i(U_i)} (x_i^{-1})^* (\rho_i \omega) = \sum_i \varepsilon_i \int_{x_i(U_i \cap V)} (x_i^{-1})^* (\rho_i \omega) \underset{\substack{\text{by Lemma above} \\ (\text{applied to } y \circ x_i^{-1})}}{=} \sum_i{}' \varepsilon \int_{y(U_i \cap V)} (y^{-1})^* (\rho_i \omega) = \varepsilon \int_{y(V)} (y^{-1})^* \omega$

Thus $(*)$ obeys 2). ∎

Cor If M oriented and $\dim M = 0$, $f \in \Omega^0_c(M)$, $\displaystyle\int_M f = \sum_{p \in M}{}' \varepsilon_p f(p)$ (ε_p given by orientation).

Def
1) If M is oriented mfd, let $-M$ denote M with the opposite orientation.
2) If M is oriented mfd, an orientation form on M is any $\omega \in \Omega^m(M)$ which induces the orientation on each fiber of TM.
3) If N is oriented mfd, and $f: M \to N$ local diffeo, the pullback orientation on M is given by $f^* \omega$ with ω orientation form of N.

Cor $\displaystyle\int_{-M} \omega = - \int_M \omega.$

Pf Exercise.