More on orientations

① M, N oriented $\Longrightarrow$ $M \times N$ oriented

$$M \times N = (-1)^{(\dim M)(\dim N)} N \times M.$$

② $f : M \to N$ $\quad Q \subset N$ $\quad f \pitchfork Q$ $\qquad P = f^{-1}(Q)$

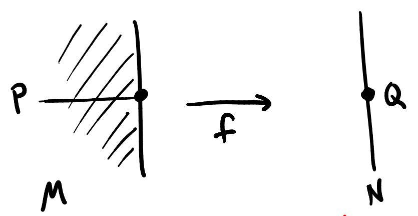
$$df : NP \xrightarrow{\sim} f^{*}NQ$$

if Q, N oriented then NQ oriented. so then NP oriented.

if also M oriented then get P oriented. $\qquad NP = TM/TP$

Ex $\quad S^2 = f^{-1}(1)$ for $f : \mathbb{A}^3 \to \mathbb{R}$ $\Rightarrow$ standard orient'ns on $\mathbb{A}^3, \mathbb{R}$ induce orient'n on S^2.

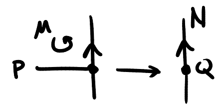
Ex



Can view ∂P two ways: $\partial(f^{-1}(Q))$ or $(\partial f)^{-1}(Q)$

Prop $\quad \partial(f^{-1}(Q)) = (-1)^{\mathrm{codim}(Q \subset N)} (\partial f)^{-1}(Q)$.

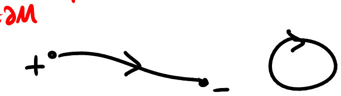
Pf Exercise. [One special case:



here get $(\partial f)^{-1}(Q) = \overset{+}{\bullet}$ $\quad \partial(f^{-1}(Q)) = \partial(\leftarrow\!\bullet) = \overset{-}{\bullet}$]

Lemma $\quad M$ oriented 1-mfd w/ ∂: $\displaystyle\sum_{p \in \partial M} \varepsilon(p) = 0$ $\qquad \varepsilon(p) = \begin{cases} +1 & p \text{ +ve} \\ -1 & p \text{ -ve} \end{cases}$

Pf Use classif'n of 1-mfds,



each has $\sum \varepsilon(p) = 0$ $\blacksquare$

Degree

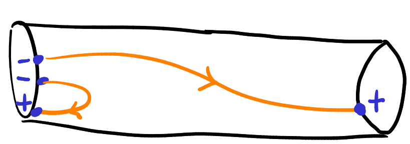
Def/Prop $\quad f : \underset{\substack{\text{compact} \\ \text{oriented}}}{M} \longrightarrow \underset{\substack{\text{connected} \\ \text{oriented}}}{N}$ $\qquad \dim M = \dim N$:

$$\deg f = \sum_{p \in f^{-1}(q)} \varepsilon(p) \quad \left[\text{where } \varepsilon = \begin{cases} +1 & \text{if } p \text{ positively oriented} \\ -1 & \text{if } p \text{ negatively oriented} \end{cases}\right], \quad q \text{ regular value}$$

This is well defined and if $f \sim g$ then $\deg f = \deg g$.

Pf As for mod 2 degree, except: we use the fact that if $F : I \times M \to N$ transverse to q

then setting $P = F^{-1}(q)$,



have $\displaystyle\sum_{p \in \partial P} \varepsilon(p) = 0$,

but also $\partial P = (-1)^{\dim M} (\partial F)^{-1}(q) = (-1)^{\dim M} \left(F_1^{-1}(q) \sqcup F_0^{-1}(q)\right) \Rightarrow \displaystyle\sum_{p \in \partial P} \varepsilon(p) = (-1)^{\dim M} \left(\deg F_1 - \deg F_0\right)$

(recall the induced orientation: $\partial(I \times M) = \{1\} \times M \sqcup \{0\} \sqcup -M$) $\blacksquare$

Prop $\underset{\substack{\text{compact}\\\text{oriented}}}{M} \xrightarrow{f} \underset{\substack{\text{compact}\\\text{connected}\\\text{oriented}}}{N} \xrightarrow{g} \underset{\substack{\text{connected}\\\text{oriented}}}{R}$: $\deg(g\circ f) = \deg f \deg g$.

Pf Pick $r \in R$ s.t. r is regular value of g and each $q \in g^{-1}(r)$ is regular value of f.

Then, use $(g\circ f)^{-1}(r) = \bigsqcup_{q\in g^{-1}(r)} f^{-1}(q)$ write ε_p^h for orient^n on p induced by h

so that
$$\deg(g\circ f) = \sum_{p\in(g\circ f)^{-1}(r)} \varepsilon_p^{g\circ f} = \sum_{q\in f^{-1}(r)} \sum_{p\in f^{-1}(q)} \varepsilon_p^{g\circ f} = \sum_{q\in f^{-1}(r)} \sum_{p\in f^{-1}(q)} \varepsilon_p^g \varepsilon_q^f$$
$$= \sum_{q\in f^{-1}(r)} \varepsilon_q^f \sum_{p\in f^{-1}(q)} \varepsilon_p^g = \sum_{q\in f^{-1}(r)} \varepsilon_q^f \deg g = (\deg f)(\deg g)$$
∎

Prop $a: S^n \to S^n$ antipodal map has $\deg(a) = (-1)^{n+1}$.
$(x^0, \dots, x^n) \mapsto (-x^0, \dots, -x^n)$

Pf $\iota: S^n \to \mathbb{A}^{n+1}$ inclusion, induced orientation on S^n. $a: \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ given by same formula, $a\circ\iota = \iota\circ a$.
Pick $p = (1,0,\dots,0) \in S^n$. Then orientation at p is given by $\omega_p = \iota^*(dx^2\wedge\cdots\wedge dx^{n+1})$ since dx^1 is +ve oriented normal at p and $dx^1\wedge\cdots\wedge dx^n$ is +ve oriented on $\mathbb{A}^n$. At $a(p)$ orientation is $\omega_{a(p)} = -\iota^*(dx^2\wedge\cdots\wedge dx^{n+1})$ instead since here $-dx^1$ is +ve oriented normal. And $a^*\iota^*(dx^2\wedge\cdots\wedge dx^n) = \iota^* a^*(dx^2\wedge\cdots\wedge dx^{n+1})$
$= (-1)^n \iota^*(dx^2\wedge\cdots\wedge dx^{n+1})$, i.e. $a^*\omega_{a(p)} = (-1)^{n+1}\omega_p$. ∎

Cor S^n admits nowhere-vanishing vector field $\iff$ n is odd.

Pf For n odd, take the vector field $\xi(x) = (-x^1, x^0, -x^3, x^2, \dots, -x^n, x^{n-1}) \in T_x S^n = \{\xi : \xi\cdot x = 0\} \subset T_x \mathbb{A}^{n+1}$
For n even, suppose we had such a vector field; then WLOG may assume $\|\xi(x)\| = 1$ $\forall x \in S^n$;
then define map $F: [0,\pi]\times S^n \to S^n$
$$(t,x) \longmapsto \underbrace{(\cos t)x + (\sin t)\xi(x)}$$
$\in S^n$ because $\|ax + b\xi(x)\|^2 = a^2\|x\|^2 + 2ab\, x\cdot\xi(x) + b^2\|\xi(x)\|^2$
$= a^2 + b^2$

This gives a homotopy between $F_0 = \text{identity}$ and $F_\pi = \text{antipodal}$

degree $= 1$ degree $= -1$

$\not\!\!\#$ ∎

Prop $\deg_2 f = \deg f \mod 2$

Rk Still, $\deg_2$ is useful when we have to deal with non-orientable manifolds.

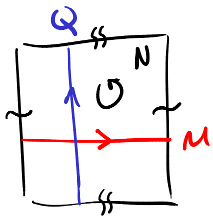
Def/Prop $f: M \to N$ $M, Q \subset N$ of complementary dim M compact oriented N oriented, Q oriented

$$I(f,Q) = \sum_{p \in \tilde{f}^{-1}(Q)} \varepsilon(p) \quad \text{where } \tilde{f} \sim f \text{ and } \tilde{f} \pitchfork Q.$$

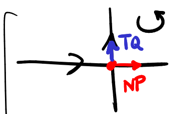
Pf well defined: again as before, now using our Lemma above.

Notation If M, N, Q as above and $M \subset N$, $f: M \hookrightarrow N$ is inclusion, then write $I(M,Q) = I(f,Q)$.
("oriented intersection number")

Ex



$I(M,Q) = +1$
$I(Q,M) = -1$

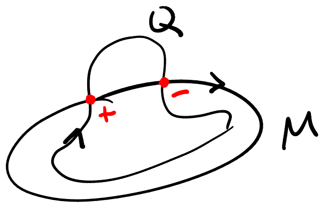


$P = f^{-1}(Q)$ gets orientation from:

put $NP \simeq TM$ first ie use

$\det TM \otimes \det TQ \xrightarrow{\sim} \det TN$
$\alpha \otimes \beta \mapsto \alpha \wedge \beta$

Ex



$I(M,Q) = 0$

Ex $M = N$, Q = oriented 0-mfd:
$$I(M,Q) = \sum_{q \in Q} \varepsilon(q).$$

Prop $I(M,Q) = (-1)^{(\dim M)(\dim Q)} I(Q,M)$ if both $M, Q \subset N$ compact.

Pf Exercise, like one you already did for I_2 (mod 2 intersection).

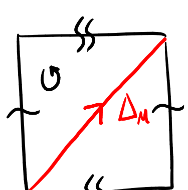
Cor $\mathbb{RP}^2$ is not orientable.

Pf $\mathbb{RP}^2$ has a submanifold $M \subset \mathbb{RP}^2$ with transverse perturbation M' s.t. $M \cap M' = pt$.
So if $\mathbb{RP}^2$ were oriented we would have $I(M,M) = \pm 1$. But $I(M,M) = (-1)^{1 \cdot 1} I(M,M)$
so $I(M,M) = 0$. $\times$ ∎

Def M compact oriented smooth mfd: ① diagonal $\Delta_M = \{(p,p) : p \in M\} \subset M \times M$
Δ_M is oriented since $\Delta_M \simeq M$ canonically.

② Euler characteristic $\chi(M) = I(\Delta_M, \Delta_M)$

Ex $M = S^1$:



$\chi(S^1) = I(\Delta_M, \Delta_M) = 0$

Ex $M = pt$: $\bullet_+$ $\chi(pt) = I(pt,pt) = 1$
$M \times M = pt$
$\Delta_M = pt$

Rk This def^n can be extended to case where M is not oriented: just use a local orientation of M near each intersection point, note the sign $\varepsilon(p)$ doesn't depend on the local orientation.

Prop $\dim M$ odd $\Rightarrow \chi(M) = 0$.

Pf $I(\Delta_M, \Delta_M) = (-1)^{(\dim M)^2} I(\Delta_M, \Delta_M) \Rightarrow I(\Delta_M, \Delta_M) = 0$. ∎