Thm (Hopf) M compact oriented, $f,g: M \to S^m$, $\deg f = \deg g \Rightarrow f \sim g$.

Thm (Jordan-Brouwer) $M \subset \mathbb{R}^n$, $\dim M = n-1$, M compact connected:

$$M \setminus N = D_0 \cup D_1 \text{ with } D_i \text{ open connected, } \overline{D_1} \text{ compact, } \partial \overline{D_1} = M.$$

A tool used in pf of both: winding number

Def ① $f: M \to \mathbb{R}^{m+1}$ smooth, M compact oriented: winding # of f around $x \in \mathbb{R}^{m-1} \setminus f(M)$

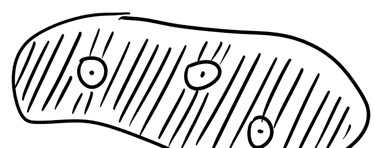
is $$\operatorname{wind}(f,x) = \deg \begin{pmatrix} M \to S^m \\ p \mapsto \frac{f(p)-x}{\|f(p)-x\|} \end{pmatrix} \in \mathbb{Z}$$

② similarly define mod 2 winding # if M not oriented

Then for Jordan-Brouwer, $D_i = \{x \in \mathbb{R}^{m-1} \setminus f(M) \mid \operatorname{wind}_2(\iota, x) = i\}$

Prop If in the above $M = \partial N$ and f extends to $F: N \to \mathbb{R}^{m+1}$ with x a regular value, then $\operatorname{wind}(f,x) = \sum_{p \in F^{-1}(x)} \varepsilon_p$; similarly for mod 2.

Pf Sketch (Similar to our analysis of Lefschetz #)



M

Let $N' = N \setminus \{\text{small balls around points } p \in F^{-1}(x)\}$

Then $\partial N' = M \cup \bigcup S^m_p$, $\deg \begin{pmatrix} \partial N' \to S^m \\ p \mapsto \frac{F(p)-x}{\|F(p)-x\|} \end{pmatrix} = 0$ since this map extends to N.

Thus we reduce to the case where $M = S^m$, $N = B^{m+1}_\varepsilon$, and $F^{-1}(x) = \{0\}$.
Deal with this case by homotopy to a linear map and using connectedness of GL_+.

Cor If $f: \mathbb{C} \to \mathbb{C}$ is a polynomial of degree n with all zeroes simple, then it has exactly n zeroes.

Pf $f(z) = z^n + g(z)$

Take a circle $S^1_M = \{|z| = M\} \subset \mathbb{C}$ s.t. on S^1_M, $|z^n| > |g(z)|$.
Clearly $f(z)$ has no zeroes with $|z| \geq M$. Same for $f_t(z) = z^n + (1-t)g(z)$.
Then $\deg(f|_{S^1_M}) = \deg(f_1|_{S^1_M}) = n$.

And all zeroes simple $\Rightarrow$ $f'(z) \neq 0$ at zeroes $\Rightarrow$ 0 is regular value of f.

Thus $n = \sum_{z \in f^{-1}(0)} \varepsilon_z$. And $\varepsilon_z = 1$ since $df_z : \underset{\mathbb{R}^2}{\underset{\shortparallel}{\mathbb{C}}} \longrightarrow \underset{\mathbb{R}^2}{\underset{\shortparallel}{\mathbb{C}}}$ is $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ if $f'(z) = a+ib$

which has $\det = a^2+b^2 > 0$.

So $\# f^{-1}(0) = n$. $\blacksquare$

Lemma $f: M \longrightarrow N$, both compact oriented, $\dim M = \dim N = n$, $\alpha \in \Omega^n(N)$: $\int_M f^*\alpha = (\deg f) \cdot \int_N \alpha$

Def ① A complex vector bundle is just like a real one, replacing $\mathbb{R}$ by $\mathbb{C}$ everywhere.

② Complexified tangent bundle $T_{\mathbb{C}}M$ has fibers $(T_{\mathbb{C}}M)_p = T_pM \otimes_{\mathbb{R}} \mathbb{C}$

③ Similarly $T^*_{\mathbb{C}}M$, $\Lambda T_{\mathbb{C}}M$, $\Lambda T^*_{\mathbb{C}}M, \dots$

If $M = \mathbb{C}$ $\quad z = x+iy$

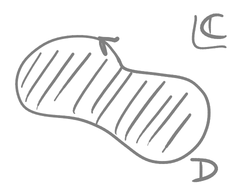
we have $dz = dx + i\,dy \in \Omega^1_{\mathbb{C}}(M)$ $\quad d\bar{z} = dx - i\,dy \in \Omega^1_{\mathbb{C}}(M)$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$\{dz, d\bar{z}\}$ and $\left\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right\}$ are dual bases $\Rightarrow$ $df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$

Then if $\frac{\partial f}{\partial \bar{z}} = 0$ (f holomorphic) then $df = \frac{\partial f}{\partial z} dz$

and $\oint_{\partial D} f\,dz = \int_D d(f\,dz) = \int_D \frac{\partial f}{\partial z} dz \wedge dz = 0$ $\quad$ [Cauchy]



Also, $\oint_{\partial D} \frac{\partial f/\partial z}{f(z)} dz = \oint_{\partial D} f^*\left(\frac{dw}{w}\right)$ $\qquad w = re^{i\theta}$

$\frac{dw}{w} = d(\log r) + i\alpha \qquad \alpha = \text{"}d\theta\text{"}$

$= i \oint_{\partial D} f^*\alpha$

$= 2\pi i \deg\left(\begin{matrix} \partial D \longrightarrow S^1 \\ z \mapsto \arg f(z) \end{matrix}\right)$

$= 2\pi i\,(\#\text{ zeroes of } f \text{ in } D)$ if f has only simple zeroes

Lemma ① $\mathcal{L}_X(\alpha\wedge\beta)=\mathcal{L}_X\alpha\wedge\beta+\alpha\wedge\mathcal{L}_X\beta \qquad \alpha,\beta\in\Omega(M)$

② $\mathcal{L}_X f = Xf \qquad f\in C^\infty(M)$

③ $d(\mathcal{L}_X\omega)=\mathcal{L}_X(d\omega) \qquad \omega\in\Omega(M)$

Def $X\in V,\ \omega\in\Lambda^k V^*$: $\iota_X\omega\in\Lambda^{k-1}V^*$ is given by $\iota_X\omega(Y_1,\dots,Y_{k-1})=\omega(X,Y_1,\dots,Y_{k-1})$

Lemma $\iota_X(\alpha\wedge\beta)=\iota_X\alpha\wedge\beta+(-1)^{|\alpha|}\alpha\wedge\iota_X\beta$

Prop $X\in\mathfrak{X}(M),\ \omega\in\Omega(M)$: $\mathcal{L}_X\omega=d(\iota_X\omega)+\iota_X(d\omega)$

Pf Say $\omega\in\Omega^k(M)$, induct on k. $k=0$: $\mathcal{L}_X f=Xf \quad \iota_X\omega=0 \quad \iota_X(df)=Xf$ ✓

$k>0$: $\omega=dx^1\wedge\alpha$ (renumber indices if needed)

$$\mathcal{L}_X\omega=\mathcal{L}_X dx^1\wedge\alpha+dx^1\wedge\mathcal{L}_X\alpha$$

$$\begin{aligned}
d(\iota_X\omega)+\iota_X d\omega &= d(\iota_X dx^1\wedge\alpha-dx^1\wedge\iota_X\alpha)-\iota_X(dx^1\wedge d\alpha)\\
&= d(\iota_X dx^1)\wedge\alpha+\cancel{\iota_X dx^1\wedge d\alpha}+dx^1\wedge d\iota_X\alpha-\cancel{\iota_X dx^1\wedge d\alpha}+dx^1\wedge\iota_X d\alpha\\
&= d(\mathcal{L}_X x^1)\wedge\alpha+dx^1\wedge\mathcal{L}_X\alpha\\
&= \mathcal{L}_X dx^1\wedge\alpha+dx^1\wedge\mathcal{L}_X\alpha
\end{aligned}$$

∎

Prop $X\in\mathfrak{X}(M)$, ϕ_t flow of X: $\exists\, Q:\Omega^k(M)\to\Omega^{k-1}(M)$ s.t. $\phi_1^*\omega-\omega=d(Q(\omega))+Q(d\omega)$

Pf Set $H_t(\omega)=\iota_X(\phi_t^*\omega)\in C^\infty(M)$

$$\begin{aligned}
\underset{\Omega^1(M)}{\frac{d}{dt}\phi_t^*\omega} &= \frac{d}{ds}\Big|_{s=0}\phi_{t+s}^*\omega\\
&= \frac{d}{ds}\Big|_{s=0}\phi_s^*\phi_t^*\omega\\
&= \mathcal{L}_X(\phi_t^*\omega)\\
&= d(\iota_X\phi_t^*\omega)+\iota_X(d\phi_t^*\omega)\\
&= dH_t(\omega)+H_t(d\omega)
\end{aligned}$$

Then set $Q(\omega)=\int_0^1 H_t(\omega)\,dt$

$$\begin{aligned}
\phi_1^*\omega-\omega &= \int_0^1\frac{d}{dt}\phi_t^*\omega\ dt\\
&= \int_0^1[dH_t(\omega)+H_t(d\omega)]\,dt\\
&= d(Q(\omega))+Q(d\omega)
\end{aligned}$$

∎

Rk ϕ_1^* is a chain homomorphism on complex $\Omega^k(M)$ and Q is a chain homotopy from ϕ_1^* to the identity.

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega^{k-1}(M) \arrow[r, "d"] \arrow[d, shift right, "1"'] \arrow[d, shift left, "\phi_1^*"] & \Omega^k(M) \arrow[r, "d"] \arrow[d, shift right, "1"'] \arrow[d, shift left, "\phi_1^*"] & \Omega^{k+1}(M) \arrow[r] \arrow[d, shift right, "1"'] \arrow[d, shift left, "\phi_1^*"] & \cdots \\
\cdots \arrow[r] & \Omega^{k-1}(M) \arrow[r, "d"] & \Omega^k(M) \arrow[r, "d"] \arrow[ul, "Q"'] & \Omega^{k+1}(M) \arrow[r] \arrow[ul, "Q"'] & \cdots \arrow[ul, dotted]
\end{tikzcd}$$

Cor In the above situation, $\phi_1^*: H^k_{dR}(M) \to H^k_{dR}(M)$ is the identity.

Pf If ω closed then $\phi_1^*\omega - \omega = d(Q(\omega))$. ■

Cor Say $f,g: M \to N$ homotopic. Then $f^* = g^*$.

Pf Say $F: M\times[0,1]\to N$ homotopy $F_0=f$, $F_1=g$. Extend arbitrarily to $M\times\mathbb{A}^1$.

Let $X = (0, \frac{\partial}{\partial t}) \in \mathfrak{X}(M\times\mathbb{A}^1)$. Flow is $\phi_t(p,s) = (p, s+t)$. $\quad \iota: M\to W$, $p \mapsto (p,0)$

Then for $\omega\in\Omega(N)$ closed, $g^*\omega - f^*\omega = \iota^*(\phi_1^*F^*\omega - F^*\omega)$ $\quad$ ($F\circ\iota = f$, $F\circ\phi_1\circ\iota = g$)

$$= \iota^*(d(Q(F^*\omega)) - Q(\cancel{d(F^*\omega)}^{\,0}))$$

$$= d(\iota^*(Q(F^*\omega)))$$

Thus $[f^*\omega] = [g^*\omega] \in H^k_{dR}(M)$. ■

Def M,N are <u>homotopy equivalent</u> if $\exists$ smooth $f: M\to N$, $g: N\to M$ s.t. $g\circ f \sim id_N$ and $f\circ g \sim id_M$.

Ex $S^n \sim \mathbb{A}^{n+1}\setminus\{0\}$, $\mathbb{A}^n\sim\{pt\}$, $B_\varepsilon(0)\sim\{pt\}$ $\quad$ (so this is <u>weaker</u> than homeo/diffeo)

Cor M,N homotopy equivalent $\Rightarrow H^k_{dR}(M)\cong H^k_{dR}(N)$.

Pf $H^k_{dR}(M) \underset{f^*}{\overset{g^*}{\rightleftarrows}} H^k_{dR}(N)$ $\quad$ $f^*\circ g^* = id_N^* = 1$

$g^*\circ f^* = id_M^* = 1$ ■

Cor (Poincare lemma) M homotopy equivalent to $\{pt\} \Rightarrow H^k_{dR}(M)\cong \begin{cases}\mathbb{R} & k=0\\ 0 & k>0\end{cases}$

(So, in this case $\omega\in\Omega^k(M)$ closed $\Rightarrow \omega\in\Omega^k(M)$ exact.)

Suppose $M\subset\mathbb{A}^{m+1}$ compact, $\dim M=m$ <u>even</u>.

<u>Gauss map</u> $\nu: M\to S^m$

$p\mapsto$ outward unit normal at p $\quad$ (<u>defined</u> using Jordan-Brouwer)

Thm (Gauss-Bonnet for hypersurfaces) $\deg\nu = \frac{1}{2}\chi(M)$.

Cor There is no embedding $\mathbb{CP}^2 \hookrightarrow \mathbb{A}^5$.

Pf of Cor $\chi(\mathbb{CP}^2)=3$ (to see this consider the map $f:\mathbb{CP}^2\to\mathbb{CP}^2$ $(z^0,z^1,z^2)\mapsto(az^0,bz^1,cz^2)$ $a,b,c\neq 1$ all distinct. Fixes $(1,0,0)$ $(0,1,0)$ $(0,0,1)$ $L(f)=3$ $f\sim id$ ■

Pf of Thm $\bar\nu: M\xrightarrow{\nu} S^m \to \mathbb{RP}^m$ (orientable!). $\deg\nu = \frac{1}{2}\deg\bar\nu$. Pick regular value $\ell\in\mathbb{RP}^m$ ie $\ell\subset\mathbb{R}^{m+1}$ line.

Fix $u\in\ell$, $\|u\|=1$. For $p\in M$ let $\xi(p)$ be the $\perp$ projection of u onto $T_pM\subset\mathbb{R}^{m+1}$

$\xi(p)=0 \iff \bar\nu(p)=\ell$. Thus, since ℓ is regular value, $\xi\in\mathfrak{X}(M)$ has finitely many zeroes.

And $\xi = u - (u\cdot\nu)\nu$ $\quad$ ($\xi: M\to\mathbb{R}^{m+1}$)

$d\xi = -(u\cdot d\nu)\nu - (u\cdot\nu)\,d\nu$ $\quad$ ($d\xi: TM\to\mathbb{R}^{m+1}$)

If $\nu(p)=u$ then, using $\nu\cdot d\nu=0$ and $\|\nu\|=1$, $d\xi_p = -d\nu_p$, thus $\det d\xi_p = (-1)^m\det d\nu_p = \det d\nu_p$

Similarly if $\nu(p)=-u$, $d\xi_p = d\nu_p$, $\det d\xi_p = \det d\nu_p$.

Thus $\mathrm{ind}_p\xi = \pm 1$ according as $d\nu$ preserves/reverses orientation. $\Rightarrow \chi(M)=\deg\bar\nu$. ■