Fermions

So far our computations have been very far from "topological": the results depended on every little detail of the input data (the function S).

To get something topological we need one key new ingredient.

We extend our 0-dimensional QFT to include anticommuting fields: ie, we replace our space $\mathcal{C}$ by some kind of "superspace," on which the space of functions is a supercommutative algebra.

$$\mathrm{Fun}(\mathcal{C}) = \mathrm{Fun}^0(\mathcal{C}) \oplus \mathrm{Fun}^1(\mathcal{C}), \quad |f| = \begin{cases} 0 \text{ if } f \in \mathrm{Fun}^0(\mathcal{C}) & (\text{"even", "bosonic"}) \\ 1 \text{ if } f \in \mathrm{Fun}^1(\mathcal{C}) & (\text{"odd", "fermionic"}) \end{cases}$$

$$fg = (-1)^{|f||g|} gf$$

Action $S \in \mathrm{Fun}^0(\mathcal{C})$.

Simplest example: $\mathcal{C} = \mathbb{R}^{0|2}$ "odd vector space"

$\mathcal{C}$ has two "coordinate functions" $\psi^1, \psi^2 \in \mathrm{Fun}^1(\mathcal{C})$ $\quad \psi^1\psi^2 \in \mathrm{Fun}^0(\mathcal{C})$

which obey $\psi^1\psi^2 = -\psi^2\psi^1, \quad (\psi^1)^2 = 0, \quad (\psi^2)^2 = 0$

So NB, $\psi^1\psi^2$ is even but $(\psi^1\psi^2)^2 = 0$!

$$\left[\mathrm{Fun}(\mathcal{C}) = \Lambda^\bullet(\mathbb{R}^{2*}) \qquad \begin{aligned} \mathrm{Fun}^0(\mathcal{C}) &= \Lambda^1((\mathbb{R}^2)^*) = (\mathbb{R}^2)^* \\ \mathrm{Fun}^1(\mathcal{C}) &= \Lambda^2((\mathbb{R}^2)^*) \end{aligned} \right]$$

The most general possible action: $S = \frac{1}{2} M \psi^1 \psi^2$

Now, we'd like to define $\quad Z = \int d\psi^1 \, d\psi^2 \, e^{-S(\psi^1, \psi^2)}$

Rule for integration over odd variables:

$$\int d\psi \, (a + b\psi) = b \qquad \left(\text{NB, this means } d(\lambda\psi^i) = \frac{1}{\lambda} d\psi^i\right)$$

$$\int d\psi^1 d\psi^2 \cdots d\psi^K \; F = \int d\psi^1 \left[\int d\psi^2 \left[\cdots \left[\int d\psi^K \; F \right] \cdots \right] \right]$$

So, expand: $Z = \int d\psi^1 d\psi^2 \left(1 - \tfrac{1}{2} M \psi^1 \psi^2\right)$

$$= -\tfrac{1}{2} M \int d\psi^1 d\psi^2 \, \psi^1 \psi^2 = -\tfrac{1}{2} M$$

[cf. the case of <u>even</u> Gaussian integral over 2 variables which gives $\frac{2\pi}{M}$]

More generally if $\mathcal{C} = \mathbb{R}^{0|2n}$, $S = \frac{1}{2} \psi^i M_{ij} \psi^j$ for an antisymmetric matrix M,

get $Z = \mathrm{Pf}(M)$

[Recall $\mathrm{Pf}(M)$ defined only for antisymm matrices, polynomial in the entries, conj. invariant, has $\mathrm{Pf}(M)^2 = \det M$ — e.g. $\mathrm{Pf}\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} = x$]

Combining bosons and fermions: take $\mathcal{C} = \mathbb{R}^{1|2}$

$\mathrm{Fun}^0(\mathcal{C}) = \mathrm{Fun}(\mathbb{R}) \oplus \mathrm{Fun}(\mathbb{R}) \psi^1 \psi^2$

$\mathrm{Fun}^1(\mathcal{C}) = \mathrm{Fun}(\mathbb{R}) \psi^1 \oplus \mathrm{Fun}(\mathbb{R}) \psi^2$

$$S(x, \psi_1, \psi_2) = S_1(x) - S_2(x) \psi^1 \psi^2$$

$$Z = \int dx \, d\psi^1 d\psi^2 \, e^{-S(x, \psi^1, \psi^2)}$$

$$= \int dx \, S_2(x) \, e^{-S_1(x)}$$

$$\langle \psi^1 \psi^2 \rangle = \int dx \, e^{-S_1(x)}$$

We could compute in perturbation theory.

But there is a special case where we can do much better:

$S_1 = \frac{1}{2} (h'(x))^2$, $S_2 = h''(x)$ for some $h : \mathbb{R} \to \mathbb{R}$

Why is this case special? It has <u>extra symmetry</u>.

Analogy: consider $\mathcal{C} = \mathbb{R}^2$, $S = f(x^2 + y^2)$. S is evidently invariant under $U(1)$ action on $\mathcal{C}$, generated by the vector field $V = y \partial_x - x \partial_y$.

A convenient notation for checking this invariance: write the "infinitesimal $U(1)$ action" as $\begin{aligned}\delta x &= \varepsilon y\\ \delta y &= -\varepsilon x\end{aligned}$. Then $\delta S = 2xf'\delta x + 2yf'\delta y = \varepsilon(2xy - 2xy)f' = 0.$

Also $dx \wedge dy$ is invariant: $\delta(dx \wedge dy) = \varepsilon(dy \wedge dy - dx \wedge dx) = 0$

So, to evaluate $Z = \int_{\mathbb{C}} dx\, dy\, e^{-S}$ we can "factor out" the direction along the $U(1)$ orbits: (except at the origin)

get $$Z = \left[\int_0^\infty r\, dr\, e^{-S}\right]\int_0^{2\pi} d\theta = 2\pi \int_0^\infty r\, dr\, e^{-S}$$

Now return to the S of our interest. S doesn't have a symmetry vector field in the usual sense. However it does have two odd symmetry vector fields:

$$V_1 = \psi^1 \frac{\partial}{\partial x} - h'(x)\frac{\partial}{\partial \psi^2}$$

$$V_2 = \psi^2 \frac{\partial}{\partial x} + h'(x)\frac{\partial}{\partial \psi^1}$$

$$V_1 S = V_2 S = 0$$

They generate a super Lie algebra: $\left([A,B] = AB - (-1)^{|A||B|}BA\right)$

$$[V_1, V_1] = -2\psi^1 h''(x)\frac{\partial}{\partial \psi^2}$$

$$[V_2, V_2] = 2\psi^2 h''(x)\frac{\partial}{\partial \psi^1}$$

$$[V_1, V_2] = \psi^1 h''(x)\frac{\partial}{\partial \psi^1} - \psi^2 h''(x)\frac{\partial}{\partial \psi^2} + 2h'(x)\frac{\partial}{\partial x}$$

Another common notation: consider infinitesimal variation of the form

$$\begin{aligned}\delta x &= \varepsilon_1 \psi^1 + \varepsilon_2 \psi^2\\ \delta \psi_1 &= \varepsilon_2 h'(x)\\ \delta \psi_2 &= -\varepsilon_1 h'(x)\end{aligned}$$

(regard $\varepsilon^1, \varepsilon^2$ as infinitesimal odd parameters)

Then $\delta S = h'(x)h''(x)(\varepsilon_1\psi^1 + \varepsilon_2\psi^2) - h''(x)(\varepsilon_2 h'(x)\psi_2 - \psi_1 \varepsilon_1 h'(x)) = 0$

The vector fields V_1, V_2 are also (super) divergence-free, i.e. they preserve the "integration measure" $dx\, d\psi_1\, d\psi_2$.

How to exploit the odd symmetries?

Heuristic:

At least formally, we could do the same as we did with even v.f. above:

in any patch away from the zeroes of (say) V_1, choose local coordinates (ψ, θ, x) such that Z becomes $\int dx\, d\psi\, e^{-S(x,\psi)} \int d\theta$

But then $\int d\theta$ just gives zero! So, do we conclude $Z=0$? Not quite: correct conclusion is that Z can be evaluated by localization — a sum over contributions from the loci where our odd vector fields vanish, i.e. where $h'=0$, $\psi_1=\psi_2=0$: critical points of h

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How to actually evaluate? Deform.

Recall for even symmetries: if V is a divergence-free vector field then $\langle V(f) \rangle = 0$

We'll use the same principle for odd symmetries.

Take $f = p'(x)\psi^1$ and

$g = (V_1+V_2) f = \psi^2 p''(x) \psi^1 + h'(x) p'(x)$

The odd symmetry $\Rightarrow$ $\langle g \rangle = 0$ [Pf: $\int dx\, d\psi^1 d\psi^2\, V_i\, g = 0$ $\forall g$ by explicit compn; in p^{tic}, $\int (V_1+V_2) f e^{-S} = \int (V_1+V_2)(fe^{-S}) = 0$]

But $\langle g \rangle = \frac{\partial}{\partial \lambda}\Big|_{\lambda=0} \int e^{-(S+\lambda g)}$ as long as $\deg(g) \leq \deg(S)$

And replacing $S \to S + \lambda g$ is equiv. to replacing $h \to h + \lambda p + O(\lambda^2)$

So: Z is invariant under deformations of h which don't change its degree!

In particular we may deform $h \to \lambda h$ and take λ large.

Recall the bosonic case: say we had $S:\mathbb{R}\to\mathbb{R}$, then expansion of $Z=\int_{-\infty}^{\infty} e^{-\lambda S(x)}$ around $\lambda=\infty$ is governed by saddle point expansion — an asymptotic expansion w/ leading term $\sum_{x_c\in P}\int_{-\infty}^{\infty} e^{-S(x_c)-S''(x_c)(x-x_c)^2}$

(where P is some nonempty subset of $\{x : S'(x)=0\}$)

Our case is slightly trickier because we aren't just rescaling the whole action by λ. Still, we can study how the critical pts behave.

$$\frac{\partial S}{\partial x} = \lambda^2 h'(x)h''(x) - \lambda h'''(x)\psi^1\psi^2, \quad \frac{\partial S}{\partial \psi^1} = -\lambda h''(x)\psi^2, \quad \frac{\partial S}{\partial \psi^2} = \lambda h''(x)\psi^1$$

Assuming $h''(x)\neq 0$, critical points are at $h'(x)=0$, $\psi^1=0$, $\psi^2=0$.

Now replace S by its quadratic approximation around each critical point: (justified by Schwarz, hep-th/9210115)

$$Z = \lim_{\lambda\to\infty} Z = \sum_c \int dx\, d\psi^1 d\psi^2\, e^{-\frac{1}{2}h''(x_c)^2(x-x_c)^2 - h''(x_c)\psi^1\psi^2}$$

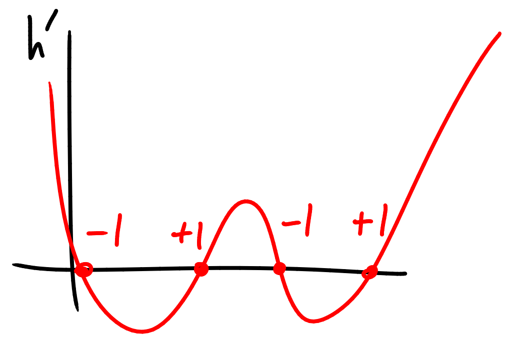
$$= \sqrt{2\pi}\sum_{c:h'(x_c)=0} \frac{h''(x_c)}{|h''(x_c)|} + \left[0 \text{ from the crit pts with } h''(x)=0\right]$$

from fermions

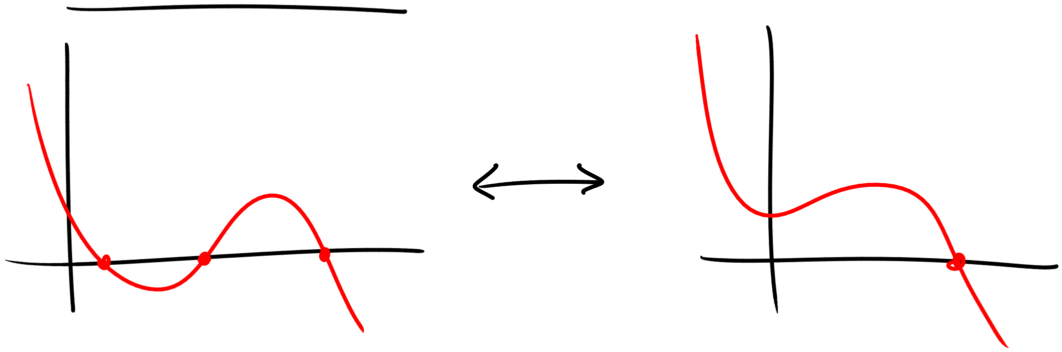
from bosons

$$= \sqrt{2\pi}\sum_c \operatorname{sgn}(h''(x_c))$$

$$\in \sqrt{2\pi}\{0,1,-1\}$$



Indeed a deformation invariant of h!



Actually, we could also see this answer by integrating out the fermions directly:

$$Z = \int_{-\infty}^{\infty} dx\ h''(x)\, e^{-\frac{1}{2}h'(x)^2} = \int dy\ e^{-\frac{1}{2}y^2} \qquad y = h'(x)$$

Limits of integration over y determine whether we get $0, +1$ or -1.

So: computation of Z localizes to fixed pts of V_1, V_2.

But $\langle f \rangle$ for $f \neq 1$ doesn't localize this way.
(Because $V_1 f \neq 0$, $V_2 f \neq 0$.)

There is a complexification of the story that is a bit richer:
take a holomorphic function $W: \mathbb{C} \to \mathbb{C}$
take $\mathcal{C} = \mathbb{C}^{1|2}$

and let $S = |W'(z)|^2 - W''(z)\psi^1\psi^2 - \overline{W''(z)}\,\overline{\psi^1}\,\overline{\psi^2}$

Invariant under 2 complex odd vector fields: $V_1 = \frac{\partial}{\partial z} + \overline{W'(z)}\frac{\partial}{\partial \psi^2} \qquad \overline{V_1} = \cdots$

$V_2 = \frac{\partial}{\partial z} - \overline{W'(z)}\frac{\partial}{\partial \psi^1} \qquad \overline{V_2} = \cdots$

Here $Z = \#$ crit. pts. of W (not counted with signs).

Also, if f is holomorphic then it's invariant under $\overline{V_1}$ and $\overline{V_2}$; this is enough to show the quadratic approx. is exact,

$$\langle f \rangle = \sum_{z_c} f(z_c)$$

Similarly if f is antihol. But for mixed f, no localization.

Let's study the holomorphic observables a bit further. Say $\overline{V} = \overline{V_1}$. Note $\overline{V}^2 = 0$.
Look at $\overline{V}$ acting on $\mathrm{Fun}^0(\mathcal{C})$. Holomorphic $f(z)$ have $\overline{V}f = 0$; and for any $g \in \mathrm{Fun}(\mathcal{C})$, $\langle \overline{V}g \rangle = 0$. Thus it's natural to consider the cohomology $R = \frac{\mathrm{Ker}\,\overline{V}}{\mathrm{Im}\,\overline{V}}$
("chiral ring")

$\mathrm{Ker}\, \bar{V} = \{\text{hol. functions } f(z)\}$

$\mathrm{Im}\, \bar{V} = \{ f(z) = W'(z) g(z) \}$

so $R = \mathbb{C}[z] / \langle W'(z) \rangle$

("Jacobi ring")