# Quantum Mechanics

Now let's move on to <u>1-dimensional</u> QFT.

1-manifold $X$ ("spacetime"): $X = \mathbb{R}, I$, or $S^1$. Parameterize by $t$.

$\mathcal{C}$ will be a function space on $X$, $S: \mathcal{C} \to \mathbb{R}$ as before.

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<u>Standard example</u>: Riemannian manifold $(Y, g)$, fix a function $V: Y \to \mathbb{R}$

Riem. metric $\eta$ on $X$

$$\mathcal{C}_X = \mathrm{Map}(X, Y),$$

$$S(\phi) = \int_X d\mathrm{vol}_X \; \|\dot\phi\|^2 + V(\phi) = \int_X dt \sqrt{\eta_{tt}} \left[ \frac{1}{2} g_{ij}(\phi)\, \dot\phi^i \dot\phi^j \eta^{tt} + V(\phi) \right]$$

$$= \int_X dt \; \tfrac{1}{2} g_{ij}(\phi)\, \dot\phi^i \dot\phi^j + V(\phi) \quad \text{if take } \eta_{tt} = 1$$

(Think of $\phi \in \mathcal{C}_X$ as possible trajectory that some <u>particle</u> could take in $Y$.)

As we did in 0-dimensional case, we may now consider <u>integrals</u>. e.g. for $X = S^1$,

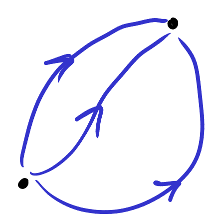
$$Z_{S^1} = \int_{\mathcal{C}_{S^1}} \mathcal{D}\phi \; e^{-S}$$

Here $\mathcal{D}\phi$ is <u>formally</u> some kind of <u>measure</u> on the $\infty$-dim space $\mathcal{C}_{S^1}$.

If $X = I$ then we should fix <u>boundary</u> <u>conditions</u>:

$$\mathcal{C}_{I_t}[y_1, y_2] = \{ \phi \in \mathrm{Map}(I, Y) : \phi(0) = y_1, \phi(t) = y_2 \}$$

$$Z_{I_t}[y_1, y_2] = \int_{\mathcal{C}_{I_t}[y_1, y_2]} \mathcal{D}\phi \; e^{-S(\phi)}$$



This is like a sum over all ways the "particle" can go from $y_1$ to $y_2$.

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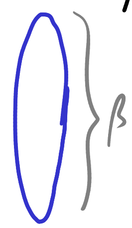
There is another, different-looking perspective on quantum mechanics: "operator formalism".

Here the basic datum is a Hilbert space $\mathcal{H}$, with a Hermitian operator $H : \mathcal{H} \to \mathcal{H}$.
("Hamiltonian")

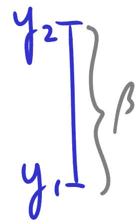
In our example, $\mathcal{H} = L^2(Y)$, $H = \frac{1}{2}\Delta + V$.

All path-integral quantities have an alternative interp. in terms of the pair $(\mathcal{H}, H)$.

e.g.: $Z_{S^1_\beta} = \mathrm{Tr}_{\mathcal{H}}(e^{-\beta H})$




$$Z_{I_\beta[y_1,y_2]} = \langle y_2 | e^{-\beta H} | y_1 \rangle$$




This formula needs some explanation:

① For any state $|\psi\rangle \in \mathcal{H}$ we let $\langle \psi|$ denote the corresponding element in $\mathcal{H}^*$, and $\langle \psi_1 | \psi_2 \rangle$ the dual pairing.

② The symbol $|y\rangle$ refers to a "$\delta$-function at $y$". This is not really an $L^2$ function, so not really an element of $\mathcal{H}$!

It is rather a distribution, characterized by $\langle y | f \rangle = f(y)$ whenever $f$ is smooth enough.

But we apply the operator $e^{-\beta H}$ to it, and that is a smoothing operator; so the result $e^{-\beta H}|y_1\rangle$ is not only $L^2$ but in fact $C^\infty$. Then $\langle y_2 | e^{-\beta H} | y_1 \rangle$ means we evaluate this $C^\infty$ function at the point $y_2$.

The quantity $\langle y_2 | e^{-\beta H} | y_1 \rangle = K_\beta(y_1, y_2)$ is sometimes called the heat kernel for the operator $H = \frac{1}{2}\Delta + V$. It is $C^\infty$ for $\beta > 0$, can be characterized by the statements

- $\left(\frac{\partial}{\partial \beta} + \frac{1}{2}\Delta_{y_1} + V(y_1)\right) K_\beta(y_1, y_2) = 0$
- $\lim_{\beta \to 0^+} K_\beta(y_1, y_2) = \delta(y_1, y_2)$

Now we give a (heuristic) "proof" of our path-$\int$ formula for $\langle y_2|e^{-\beta H}|y_1\rangle$:

It will be convenient to <u>expand</u> on the "basis" of $\delta$-functions.

Recall that in a fin-dim vector space $V$ we may write the identity operator wrt a basis $\{e_i\}$ as

$$1 = \sum_{i=1}^{n} e_i \otimes e_i^* \in V \otimes V^* = \mathrm{Hom}(V,V)$$

Here, similarly we write $1 = \int dy \, |y\rangle\langle y|$

This looks <u>formal</u> since $|y\rangle$ is only distributional. But when applied to smooth enough functions, it will be OK.

Fix some $N \geq 1$. Let $y^{(0)} = y_1$, $y^{(N)} = y_2$. We're going to break the interval $I_\beta$ into $N$ pieces of length $\Delta t = \beta/N$.

$$\langle y_2|e^{-\beta H}|y_1\rangle = \langle y_2|\underbrace{e^{-H\Delta t} \cdot e^{-H\Delta t} \cdots e^{-H\Delta t}}_{N \text{ times}}|y_1\rangle$$

$$= \int dy^{(1)} \cdots dy^{(N-1)} \langle y^{(N)}|e^{-H\Delta t}|y^{(N-1)}\rangle \langle y^{(N-1)}|e^{-H\Delta t}|y^{(N-2)}\rangle \cdots \langle y^{(1)}|e^{-H\Delta t}|y^{(0)}\rangle$$

$$= \int dy^{(1)} \cdots dy^{(N-1)} \prod_{n=0}^{N-1} \langle y^{(n+1)}|e^{-H\Delta t}|y^{(n)}\rangle$$

$$= \int dy^{(1)} \cdots dy^{(N-1)} \, K_{\Delta t}(y^{(n+1)}, y^{(n)})$$

(built from local inv'ts of metric)

(Berline-Getzler-Vergne)

Now use the short time asymptotics: as $\Delta t \to 0$, $K_{\Delta t}(y,y') \sim \left(\frac{1}{2\pi\Delta t}\right)^{\frac{1}{2}\dim Y} a(y) \exp\left[-\frac{1}{2\Delta t} d(y,y')^2\right]$

$$\sim \int dy^{(1)} \cdots dy^{(N-1)} \prod_{n=0}^{N-1} \left(\frac{1}{2\pi\Delta t}\right)^{\frac{1}{2}\dim Y} a(y^{(1)}) \cdots a(y^{(n)}) \exp\left[-\frac{1}{2\Delta t} d(y^{(n+1)}, y^{(n)})^2\right]$$

$$\sim \int dy^{(1)} \cdots dy^{(N-1)} \prod_{n=0}^{N-1} \left(\frac{1}{2\pi\Delta t}\right)^{\frac{1}{2}\dim Y} a(y^{(1)}) \cdots a(y^{(n)}) \exp\left[-\frac{\Delta t}{2}\left(\frac{d(y^{(n+1)}, y^{(n)})}{\Delta t}\right)^2\right]$$

Now consider $y^{(n)}$ as functions on $\mathcal{C}_{I_t[y_1,y_2]}$: namely $y^{(n)}(\varphi) = \varphi(n\Delta t)$.

Then <u>formally</u> we may imagine that as $N \to \infty$, $\left(\frac{1}{2\pi\Delta t}\right)^{\frac{N}{2}\dim Y} dy^{(1)} \cdots dy^{(N-1)} \, a(y^{(1)}) \cdots a(y^{(N-1)})$

<u>converges</u> to a measure $\mathcal{D}\phi$ on $\mathcal{C}_{I_t[y_1,y_2]}$. The exponential, meanwhile,

has $\dfrac{d(y^{(n+1)}, y^{(n)})}{\Delta t} \longrightarrow \|\dot\phi(n\Delta t)\|$, so we get a Riemann sum; as $N \to \infty$

we get altogether $\displaystyle\int_{\mathcal{C}_{I_\beta}[y_1,y_2]} \mathcal{D}\phi \ \exp\left[-\int dt \tfrac{1}{2}\|\dot\phi(t)\|^2\right]$ as desired.

(Remark: although the "measure" $\mathcal{D}\phi$ doesn't rigorously $\exists$, the full integrand $\mathcal{D}\phi\, e^{-S}$ does: it is "Wiener measure.")

For general $V$ it's similar, using "Trotter product formula" $e^{A+B} = \lim_{N\to\infty}\left[e^{\frac{A}{N}} e^{\frac{B}{N}}\right]^N$, applied to $A = \beta\Delta$, $B = \beta V$.

A similar argument also establishes (formally) $Z_{S^1_\beta} = \mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H}$.

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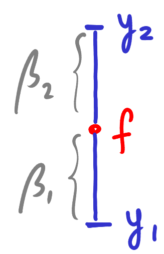
Local observables: given any $f: M \to \mathbb{R}$ (compactly supported, say) "mult. by $f$" gives an operator $f: \mathcal{H} \to \mathcal{H}$. Then, similarly to the above, we get

$$\langle y_2 | e^{-\beta_1 H} f\, e^{-\beta_2 H} | y_1 \rangle = \int_{\mathcal{C}_{I_{\beta_1+\beta_2}}[y_1,y_2]} \mathcal{D}\phi \ f(\phi(\beta_1))\, e^{-S(\phi)}$$

Picture of this:




(This observable is "local" in the sense that it only involves the fields at one time.)

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