Supersymmetric gauge theory

We are now going to write a theory which extends the Yang–Mills action. $SU(2)_R$

First, do it just for $X = \mathbb{R}^4$. Fix an auxiliary "R-symmetry" vectorspace $R \simeq \mathbb{C}^2$

$$\mathcal{C} = \begin{cases} (P, \nabla) : \text{principal } G\text{-bundle w/ conn over } X \\ \lambda^{\pm} \in \Gamma(S^{\pm} \otimes \mathfrak{g}_{\mathbb{C},P} \otimes R) \\ \phi \in \Gamma(\mathfrak{g}_{\mathbb{C},P}) \\ D \in \Gamma(\mathfrak{g}_{\mathbb{C},P} \otimes \mathrm{Sym}^2(R)) \end{cases}$$

Wrt a basis of R, expand $\lambda^{\pm}$ as pair $\lambda_1^{\pm}, \lambda_2^{\pm} \in \Gamma(S^{\pm} \otimes \mathfrak{g}_{\mathbb{C},P})$, also expand D as triplet D_{11}, D_{12}, D_{22}; let δ denote Herm. metric in R, $\varepsilon \in \Lambda^2(R)$ of unit norm.

$$S = \frac{1}{g^2} \int_X \mathrm{Tr}\Big(\overbrace{-\tfrac{1}{4} F \wedge * F + \nabla_\mu \bar{\phi} \nabla^\mu \phi - i \delta^{vw} \langle \lambda_v^-, \nabla\!\!\!/\, \lambda_w^+ \rangle}^{\text{usual minimally coupled kinetic terms}}$$
$$+ \underbrace{\tfrac{1}{4} \delta^{vv'} \delta^{ww'} D_{vw} D_{v'w'}}_{\text{auxiliary field}} \underbrace{- \tfrac{1}{2} [\phi, \bar{\phi}]^2}_{\text{potential}} \underbrace{- i\sqrt{2}\, \varepsilon^{vw} \langle \lambda_v^-, [\bar{\phi}, \lambda_w^-] \rangle + i\sqrt{2}\, \varepsilon^{vw} \langle \lambda_v^+, [\phi, \lambda_w^+] \rangle}_{\text{"Yukawa" couplings}} \Big)$$
$$+ i \frac{\vartheta}{4\pi^2} \int_X \mathrm{Tr}(F \wedge F)$$

(If we analytically continued to Minkowski signature and put $\lambda^+ = \bar{\lambda}^-$ then this would be naturally real.) Note D enters quadratically and can be integrated out for free ("auxiliary field") but it's convenient to keep it around, as we'll see.

This action has a lot of symmetries:

- gauge symmetry gp. $\mathcal{G}$
- Poincaré symmetry $I\,Spin(4)$

also possessed by ordinary gauge theory

$\cup$ translation vector fields P_v for $v \in (\mathbb{R}^4)$

- "R-symmetry" $SU(2)$ acting on $\lambda^{\pm}$ and D (via its action on R)
- "R-symmetry" $U(1)$ acting by $\lambda^{\pm} \to e^{\mp i\theta} \lambda^{\pm}$, $\varphi \to e^{2i\theta} \varphi$
- odd symmetries: vector fields Q_ξ for $\xi \in (S^+ \oplus S^-) \otimes R$

$\Longrightarrow$ for an infinitesimal param. ξ we get inf^l variations,

$$\delta\phi = \sqrt{2}\,\varepsilon^{vw}\langle \xi^+_v, \lambda^+_w\rangle$$

$$\delta A_\mu = \delta^{vw}\left(i\langle \xi^+_v, \sigma_\mu \lambda^-_w\rangle - i\langle \lambda^+_v, \sigma_\mu \xi^-_w\rangle\right)$$

$$\delta\lambda^\pm_{v'} = \delta^{vw} D_{vv'}\xi^\pm_w - i\xi^\pm_{v'}[\phi,\bar\phi] \mp i[\sigma_\mu,\sigma_\nu]\,\xi^\pm_{v'}F^{\mu\nu}$$
$$\pm i\sqrt{2}\,\varepsilon_{v'w}\sigma^\mu \xi^\mp_w D_\mu\phi$$

$$\delta D_{vw} = i\langle \xi^-_v, \not{D}\lambda^+_w\rangle + i\sqrt{2}\,\xi^+_v[\lambda^+_w, \bar\phi] + i\sqrt{2}\,\xi^-_v[\lambda^-_w, \phi] + (v \leftrightarrow w)$$

σ_μ = Clifford action of $\partial/\partial x^\mu$

$F = F_{\mu\nu}\,dx^\mu \wedge dx^\nu$

These vector fields have $\{Q_\xi, Q_{\xi'}\} = P_{T(\xi,\xi')}$ ("anticommutator of supersymmetries is translation")

where T is a map of $\underline{\mathrm{Spin}(4)\text{ reps}}$,

$$T : \left((S^+\oplus S^-)\otimes R\right) \otimes \left((S^+\oplus S^-)\otimes R\right) \longrightarrow V$$

[V = fund^l rep of $SO(4)$]

[induced from $S^+\otimes \bar S^- \to V$ and $R\otimes\bar R \to V$
$S^-\otimes \bar S^+ \to V$]

Because of these <u>odd symmetries</u> we will expect some nice <u>localization</u> for computing invariant observables. <u>For example</u>:

$\delta\lambda^\pm_{v'} = 0$ would say

$$\delta^{vw} D_{vv'}\xi^\pm_w - i\xi^\pm_{v'}[\phi,\bar\phi] \mp i[\sigma_\mu,\sigma_\nu]\xi^\pm_{v'}F^{\mu\nu} \pm i\sqrt{2}\,\varepsilon_{v'w}\sigma^\mu\xi^\mp_w D_\mu\phi = 0$$

If we set $D_{vv'} = 0$ (as we should if we're interested in <u>minima</u> of S)

and also suppose $[\phi,\bar\phi] = \nabla\phi = 0$

then this says $\not{F}\xi^\pm_{v'} = 0$ $\quad$ ($\not{F} = [\sigma_\mu,\sigma_\nu]F^{\mu\nu}$, $\not{F} : S^\pm \to S^\pm$)

(i.e. $\not{F}(x)\,\xi^\pm_{v'} = 0 \quad \forall x \in \mathbb{R}^4$)

Now we can ask: for which F does $\not{F}$ annihilate some elt $\xi \in S^+ \oplus S^-$?

Answer: this happens only if F is either self-dual or anti-self-dual!

[Because:

$$\begin{array}{ccc} F & \longmapsto & \not{F} \\ \Lambda^2(\mathbb{R}^4) & & so(4) \\ \| & \longrightarrow & \| \\ \Lambda^{2,+} \oplus \Lambda^{2,-} & & su(2) \times su(2) \end{array}$$

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So, if we compute an observable that is <u>annihilated</u> by some Q_{ξ^+} (Q_{ξ^-}) we'd expect localization to moduli space of instantons (anti-instantons) on $\mathbb{R}^4$.

"Nekrasov function" is of this sort, for a cleverly chosen observable...

But our interest now is in computing on some compact X, <u>not</u> on $\mathbb{R}^4$.

For this, we'll need to make a non-obvious modif^n of the action...