Twisting

We can put our SUSY gauge theory on general Riemannian X.
$ISO(4)$ replaced by $\mathrm{Isom}(X)$. ($ISpin(4)$ repl. by gp of spin isometries.)

But, now ask: what are the analogues of the odd vector fields Q_ξ?
For any $\xi \in \Gamma(S_B^+ \oplus S_B^-)$ we can write an odd v.f., but it annihilates the action S only if $\nabla \xi = 0$. What can we do on an arbitrary X, maybe w/ no isometries?

Idea of twisting: replace the "R-symmetry vector space" R by an $SU(2)$ vector bundle over X, with a fixed (non-dynamical) connection.

To get this bundle: fix a homomorphism $\iota : Spin(4) \to SU(2)$
Using ι, the rep. R of $SU(2)$ induces a rep. R^ι of $Spin(4)$.

Now, we write exactly the same action we wrote before, just replacing $R \mapsto (R^\iota)_B$ with B the $Spin(4)$-bundle over X given by the spin structure, and using covariant derivatives where needed.

Our new field space:

$$\mathcal{C}_X = \begin{cases} (P, \nabla) : \text{principal } G\text{-bundle w/ conn over } X \\ \lambda^\pm \in \Gamma((S^\pm \otimes R^\iota)_B \otimes \mathfrak{g}_{\mathbb{C},P}) \\ \phi \in \Gamma(\mathfrak{g}_{\mathbb{C},P}) \\ D \in \Gamma(\mathfrak{g}_P \otimes \mathrm{Sym}^2(R^\iota)_B) \end{cases}$$

Let's choose: $\iota : SU(2)_+ \times SU(2)_- \to SU(2)$
$\qquad\qquad (g_+, g_-) \mapsto g_+$

Then $R^\iota = S^+$.
So really the effect of twisting is "replace R with S^+ everywhere."

In particular, now look at our odd vector fields. They were generated by $\xi \in \Gamma((S_B^+ \oplus S_B^-) \otimes R)$.
Twist replaces that with $\xi \in \Gamma((S^+ \otimes S^+ \oplus S^- \otimes S^+)_B)$.

$$= \Gamma\left((\mathbb{C} \oplus \mathrm{Sym}^2(S^+) \oplus \mathrm{fund})_B \right)$$

$$\left[= \Gamma\left(((1,1) \oplus (3,1) \oplus (2,2))_B \right) \quad \text{in physicists' notation} \right]$$

Note, one trivial summand! This trivial bundle does have a c.c. section, no matter what X is.

$\Rightarrow$ the twisted version of the theory has a single odd vector field Q on $\mathcal{E}$, $QS = 0$.

Also, one summand $\simeq TX$, giving v.f. Q_v for $v \in \Gamma(TX)$. (But $Q_v S \neq 0$ generally)

Let's write the action now, in twisted notation:

$$\mathcal{E} = \begin{cases} (P, \nabla): \text{ principal } G\text{-bundle w/ conn over } X \\ \psi \in \Gamma(\mathfrak{g}_{\mathbb{C},P} \otimes \mathrm{fund}_B) \\ \eta \in \Gamma(\mathfrak{g}_{\mathbb{C},P}) \\ \chi \in \Gamma(\mathfrak{g}_{\mathbb{C},P} \otimes \mathrm{Sym}^2(S^+)) \\ \phi \in \Gamma(\mathfrak{g}_{\mathbb{C},P}) \\ D \in \Gamma(\mathfrak{g}_P \otimes \mathrm{Sym}^2(S^+)) \end{cases}$$

[NB: X does not need to be spin for this!]

acts on one of the two S^- factors, takes $(S^-)^2 \to$ fund

$$S = \frac{1}{g^2} \int_X \mathrm{Tr}\Big(\|\nabla\phi\|^2 - i\langle \psi, \not{\nabla}\chi \rangle - i\psi^\mu \nabla_\mu \eta - \frac{1}{4}\|F\|^2$$
$$+ \frac{1}{4}\|D\|^2 - \frac{1}{2}[\phi, \phi^\dagger]^2 - \frac{i}{\sqrt{2}} \langle \chi, [\phi, \chi] \rangle$$
$$+ i\sqrt{2}\, \eta[\phi, \eta] - \frac{i}{\sqrt{2}} \langle \psi, [\psi, \bar{\phi}] \rangle \Big)$$

The odd vector field acts by:

[not a contradiction as this is a complex vector field — like $\frac{\partial}{\partial z}$ which has $\frac{\partial}{\partial z}(z) = 1$, $\frac{\partial}{\partial z}(\bar{z}) = 0$]

$$\delta\phi = 0 \qquad \delta\bar{\phi} = \varepsilon 2\sqrt{2}\, i\eta$$
$$\delta A = \varepsilon\psi \qquad \delta\chi = i\varepsilon(F^+ - D)$$
$$\delta\eta = \varepsilon[\phi, \bar{\phi}] \qquad \delta D = \varepsilon(2\nabla\psi)_+ + 2\sqrt{2}\,\varepsilon[\phi, \chi]$$
$$\delta\psi_\mu = \varepsilon 2\sqrt{2}\, \nabla_\mu \phi$$

Fixed points: $\quad F^+ = D, \ \nabla\phi = 0, \ [\phi, \bar{\phi}] = 0.$

But crit pts have $D = 0$ so we expect localiz^n to $F^+ = 0, \ \nabla\phi = 0$ for any observables that are annihilated by δ.

What does this eq. $\nabla\phi = 0$, $[\phi,\bar\phi]=0$ mean?

If it has a solution then we have an inf^{l} automorphism of (P,∇).

Decompose fund$_p$ under the action of ϕ: $[\phi,\bar\phi]=0 \Rightarrow$ each fiber splits into $\oplus$ of 2 lines with opposite eigenvalues $\lambda, -\lambda$. $\nabla\phi = 0 \Rightarrow \lambda$ is constant $\Rightarrow$ we get a global decomposition $(P,\nabla) = (\mathcal{L},\nabla')\oplus(\mathcal{L}^{-1},\nabla'')$ of two $U(1)$ bundles w/connection.

("reducible connection".)

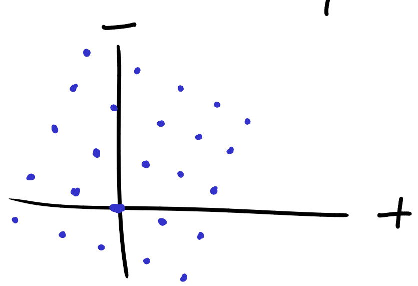
If also $F^+ = 0$ then (P,∇) is a reducible instanton.

Source of headaches in Donaldson theory since they have larger-than-usual stabilizer in $\mathcal{G}$ $\Rightarrow$ give singularities on the moduli space of instantons!

We'd like to avoid these problems. NB: if we have a reducible instanton then $(\mathcal{L},\nabla')$ is a $U(1)$ instanton. $F_{\nabla'} \in \Omega^{2,-}(X)$

$$\Rightarrow \frac{1}{2\pi}[F_{\nabla'}] \in H^{2,-}(X) \cap H^2(X,\mathbb{Z})$$

The existence of any nonzero elt in $H^{2,-}(X)\cap H^2(X,\mathbb{Z})$ is a constraint on (X,g):



as long as $b_2^+(X)\geq 1$, a "generic" lattice would miss $H^{2,-}(X)$.

So, let's suppose $b_2^+(X)\geq 1$ and g is "generic." Then we would expect localiz^n to the space of sol^{n}s of $F^+=0$, modulo gauge.

What are the good observables? Need $\mathcal{O}$ with $Q(\mathcal{O})=0$.

Simplest: $\mathcal{O}^{(0)}(x) = \mathrm{Tr}\,\phi^2(x)$. Indeed has $Q(\mathcal{O}^{(0)}(x))=0$.

To get others: use the odd v.f. G_v assoc. to vector fields v on X,

$$\delta\phi = \frac{1}{2\sqrt{2}}\langle v,\phi\rangle \qquad \delta\bar{\phi} = 0$$

$$\delta A_\mu = \frac{i}{2}(g_{\mu\nu}\eta - i\chi_{\mu\nu})v^\nu \qquad \delta\chi = -\frac{3i\sqrt{2}}{8} * \nabla_v\bar{\phi}$$

$$\delta\eta = -\frac{i\sqrt{2}}{4}\nabla_v\bar{\phi} \qquad \delta D = -\frac{3i}{4} * \nabla_v\eta + \frac{3i}{2}\nabla_v\chi$$

$$\delta\psi_\mu = -(F^-_{\mu\nu} + D_{\mu\nu})v^\nu$$

It obeys $[Q, G_v] = P_v$

So: if we define "descent" by $\mathcal{O}^{(k+1)}(x) = \sum_\mu G_{\frac{\partial}{\partial x^\mu}}\mathcal{O}^{(k)}(x) \wedge dx^\mu$

then $Q\left(\mathcal{O}^{(1)}(x)\right) = \frac{\partial}{\partial x^\mu}\mathcal{O}^{(0)}(x)\, dx^\mu = d\mathcal{O}^{(0)}(x)$

similarly, $Q(\mathcal{O}^{(k)}(x)) = k\, d\mathcal{O}^{(k-1)}(x)$

Hence, if γ is a k-cycle on X, and we define $\mathcal{O}^{(k)}(\gamma) = \int_\gamma \mathcal{O}^{(k)}(x)$

then we have $Q(\mathcal{O}^{(k)}(\gamma)) = 0$

We'll only use $k = 0,1,2$.

So, we expect localiz^n for observables of the form

$$\langle \mathcal{O}^{(0)}(x_1)\,\mathcal{O}^{(0)}(x_2)\cdots\mathcal{O}^{(1)}(\delta_1)\,\mathcal{O}^{(1)}(\delta_2)\cdots\mathcal{O}^{(2)}(S_1)\,\mathcal{O}^{(2)}(S_2)\cdots\rangle$$

Deformation invariance

As we had in previous examples, we expect $\langle Q\mathcal{O}\rangle = 0$ for any $\mathcal{O}$

In particular: $\langle\mathcal{O}^{(0)}(x_1)\rangle - \langle\mathcal{O}^{(0)}(x_2)\rangle$

$$= \int_{x_1}^{x_2}\langle Q(\mathcal{O}^{(1)}(x))\rangle = 0$$

So $\langle\mathcal{O}^{(0)}(x)\rangle$ should be <u>independent of x</u>: just write it $\langle\mathcal{O}^{(0)}\rangle$

In a similar way, $\langle \mathcal{O}^{(k)}(\gamma) \rangle$ should depend only on the homology class $[\gamma] \in H_k(X, \mathbb{Z})$

Also, the whole action has the nice property

$$S = \frac{1}{g^2}\left[Q(V) - \frac{1}{2}\int_X \mathrm{Tr}\, F \wedge F\right] + i\frac{\vartheta}{4\pi^2}\int_X \mathrm{Tr}\, F \wedge F$$

$$= \frac{1}{g^2} Q(V) + \frac{i\tau}{4\pi}\int \mathrm{Tr}\, F \wedge F \qquad \left[\tau = \frac{\vartheta}{\pi} + \frac{i}{2g^2}\right]$$

where $V = \int \mathrm{Tr}\left(\frac{i}{4}\langle \chi, F + D \rangle - \frac{1}{2}\eta[\phi, \bar{\phi}] + \frac{1}{2\sqrt{2}}\psi \nabla \bar{\phi}\right)$

In particular, all the metric dependence is in the term $Q(V)$!

So, at least formally we expect that all Q-invariant correl. func. are indep of metric on X.
i.e. this is a "topological quantum field theory," in physicists' sense.

Very different from the usual kind of field theory.