Computing the observables

To compute $\langle \dots \rangle$: $\quad \mathcal{M} = \{ (P,\nabla) : F^+ = 0 \} / \mathcal{G} \subset \mathcal{C}/\mathcal{G}$

Consider the limit $g \to 0$. Semiclassical expansion around $\mathcal{M}$ should be exact.

Pick a point (P,∇) of $\mathcal{M}$ to expand around, corresponding to an irreducible and regular connection.
("regular": $\operatorname{coker} d^+ = 0$)

$$\underset{\eta \;\; \phi}{\Omega^0(\mathfrak{g}_{\mathbb{C},P})} \oplus \underset{\chi \;\; D}{\Omega^2(\mathfrak{g}_{\mathbb{C},P})} \underset{L^*}{\overset{L}{\rightleftarrows}} \underset{\psi \;\; \delta A}{\Omega^1(\mathfrak{g}_{\mathbb{C},P})}$$

Writing $L = \nabla + (d^+_\nabla)^*$, $f = \eta + \psi + \chi$, the fermion kinetic term is $\langle f, Lf \rangle$

$\Delta = LL^* + L^*L$, $b = \phi + \delta A + D$, the boson kinetic term is $\langle b, \Delta b \rangle$ (after gauge fixing!)

∇ irred regular $\Rightarrow$ only zero eigenvals. in fermion kin. term are $\psi \in \ker \nabla^* \cap \ker d^+_\nabla$
zero eigenvals. in boson kin. term are $\delta A \in$ " "

The zero eigenvals. in boson directions are tangent directions to $\mathcal{M}$.
(to see they are at least formal tangent dirs: $F^+_{\nabla + \varepsilon A} = F^+_\Delta + \varepsilon (d_\nabla A)^+ + O(\varepsilon^2) = 0$ and $\nabla_A^* A = 0$ is gauge fixing)
But to really prove they are tangent dir requires work — see e.g. Donaldson-Kronheimer

The zero eigenvals. in fermion directions thus make up fiber of $\Pi T\mathcal{M}$.
Thus, in the limit $g \to 0$, we'd expect localization on $\Pi T\mathcal{M}$.
This is much like what we had in SUSY QM where we localized on ΠTY:
expand around any point of $\Pi T\mathcal{M}$ to quadratic order, and then perform the Gaussian integrals in the normal directions to $T\mathcal{M} \subset \mathcal{E}$. The result will be a density on $\Pi T\mathcal{M}$ i.e. a differential form on $\mathcal{M}$.

If we expand around a point with $\psi = 0$ then the quadratic part is given by the kinetic terms we wrote above. Same operators for bosons and fermions $\Rightarrow$ these determinants cancel one another (perhaps up to a tricky sign — overlook that for now).

If we expand around $\Psi \neq 0$ there is <u>one</u> new term in the quadratic exp^n:

$$\mathrm{Tr}\left(\phi\left[\Psi_\mu, \Psi^\mu\right]\right)$$

So, in computing (say) $\langle \mathcal{O}^{(0)} \rangle$ we have to do Gaussian integral

$$I(\Psi) = \int \mathcal{D}\phi \ \mathrm{Tr}(\phi(x)^2) \ e^{-\int_X \mathrm{Tr}\left(\|D\phi\|^2 - \frac{i}{\sqrt{2}}\bar{\phi}[\Psi_\mu, \Psi^\mu]\right)}$$

Since it's <u>Gaussian</u>, it's straightforward, at least in principle.
To do it: expand the exponential, note that <u>only</u> the term with two $\bar{\phi}$ in it actually contributes (all others vanish by R-symmetry $\phi \to e^{2i\vartheta}\phi$)

$$\text{So } I(\Psi) = \int \mathcal{D}\phi \ \mathrm{Tr}\,\phi(x)^2 \left[-\frac{i}{\sqrt{2}}\int dy \ \mathrm{Tr}\,\bar{\phi}(y)[\Psi_\mu(y), \Psi^\mu(y)]\right]^2 e^{-\int_X \mathrm{Tr}\,\|\nabla\phi\|^2}$$

$$= \mathrm{Tr}\left(H(x)^2\right) \text{ where } H(x) = -\frac{i}{\sqrt{2}}\int_X d^4y \ G(x,y)\left[\Psi_\mu(y), \Psi^\mu(y)\right]$$

$G =$ Green's f^n for $\nabla^*\nabla$

Expanding Ψ wrt a basis of $\ker L^*$, $\Psi = \sum_i \varepsilon_n f_n$ [ε_n odd, $f_n \in \ker L^*$],

$I(\Psi)$ is <u>quartic</u> in the ε_n

$\Longrightarrow I(\Psi)$ is a <u>4-form</u> on $\mathcal{M}$; call this $\Phi^{(0)}$

In a similar way the operators $\mathcal{O}^{(k)}(\gamma)$ give <u>$(4-k)$-forms</u> $\Phi^{(k)}(\gamma)$ on $\mathcal{M}$

Then performing the remaining integral over $\Pi T\mathcal{M}$ we obtain

$$\left\langle \prod \mathcal{O}^{(k_i)}(\gamma_i) \right\rangle = \int_{\mathcal{M}} \prod \Phi^{(k_i)}(\gamma_i)$$