Complex Geometry: Exercise Set 2

Exercise 1

1. Show that the holomorphic tangent bundle of $\mathbb{CP}^1$ is isomorphic (as a holomorphic line bundle) to $\mathcal{O}(2)$.
2. (For those who know about Lie algebras.) From this and a previous exercise, it follows that the space of holomorphic vector fields on $\mathbb{CP}^1$ is 3-dimensional. Taking brackets we thus obtain a complex 3-dimensional Lie algebra. Write out the Lie algebra structure explicitly. Show that this Lie algebra is isomorphic to $sl(2,\mathbb{C})$. (This reflects the fact that the group $SL(2,\mathbb{C})$ acts holomorphically on $\mathbb{CP}^1$.)

Exercise 2

1. Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are two holomorphic line bundles on a complex manifold X, of dimension at least 2. Suppose that for some point $x \in X$, $\mathcal{L}_1|_{X\setminus\{x\}} \simeq \mathcal{L}_2|_{X\setminus\{x\}}$. Show that $\mathcal{L}_1 \simeq \mathcal{L}_2$. (You will probably need Hartogs' Theorem, Proposition 1.1.4 of Huybrechts.)
2. Show by example that the same is not true if X is of dimension 1.

Exercise 3

Suppose (X, I) is an almost complex manifold. The *Nijenhuis tensor* N is a section of $(T^*X)^{\otimes 2} \otimes TX = \mathrm{Hom}(TX^{\otimes 2}, TX)$, given by

$$N(v,w) = [v,w] + I[Iv,w] + I[v,Iw] - [Iv,Iw]$$

for two vector fields v, w on X.

1. Show that the above formula indeed defines a tensor, i.e. $N(v,w)$ at a point $x \in X$ only depends on the values $v(x)$ and $w(x)$, not on their extension to vector fields on X; this amounts to checking that $N(fv,w) = fN(v,w)$ and $N(v,fw) = fN(v,w)$ for any function f on X.
2. Show that $N = 0$ if I is induced from an actual complex structure on X. (The converse is also true, but it will be easier to prove this after the next lecture.)

Exercise 4

Suppose (X, I) is an almost complex manifold, and $\alpha \in \Omega^{p,q}(X)$. Show that

$$\mathrm{d}\alpha \in \Omega^{p-1,q+2}(X) \oplus \Omega^{p,q+1}(X) \oplus \Omega^{p+1,q}(X) \oplus \Omega^{p+2,q-1}(X).$$