Complex Geometry: Exercise Set 4

Exercise 1

1. Suppose M is an oriented Riemannian manifold of dimension n. Verify the assertion from class that $\star^2 = (-1)^{k(n-k)}$ acting on $\Omega^k(M)$.
2. If $M = X$ is complex, show that $\star^2 = (-1)^k$ acting on $\Omega^k(X)$.

Exercise 2

Directly verify two assertions from class:

1. If $M = \mathbb{R}^n$ with its usual flat metric, the Laplacian Δ acting on differential forms simply acts by $\Delta(\sum_I f_I dx_I) = \sum_I \Delta(f_I) dx_I$ where the Δ on the right is the usual Laplacian acting on functions.
2. If $X = \mathbb{C}^n$ with its usual flat metric, then $\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta$.

Exercise 3

Suppose X is Kähler and α is a closed $(1,1)$-form which is primitive (i.e. $\Lambda(\alpha) = 0$). Show that $\Delta\alpha = 0$.

Exercise 4

1. Suppose V is a vector space with compatible inner product, complex structure and fundamental form (g, I, ω). Suppose $W \subset V$ is a subspace of dimension $2m$. Choose an orientation on W; together with g this induces a volume form vol_W. Show that $\mathrm{vol}_W / \omega^m|_W \geq \frac{1}{m!}$, with equality if and only if W is a complex subspace of V, i.e. if $IW = W$.
2. Suppose X is a Kähler manifold and Y a compact submanifold of dimension $2m$. Show that $\mathrm{vol}(Y) \geq \int_Y \frac{\omega^m}{m!}$, with equality if and only if Y is a complex submanifold of X.
3. Suppose X is a Kähler manifold for which ω is exact ($\omega = d\alpha$ for some α). Show that X has no compact complex submanifolds (in particular X is not compact).

Exercise 5

1. Verify by hand that the Kähler identities hold on $\mathbb{C}^m$.
2. Verify that the Kähler identities *do not* hold for the Hermitian metric on $\mathbb{C}^2$ with fundamental form $\omega = i dz_1 \wedge d\bar{z}_1 + i(|z_1|^2 + 1) dz_2 \wedge d\bar{z}_2$. (For example, try computing $[\partial, L]$.)

Exercise 6

Suppose X is a compact Kähler manifold, of dimension n.

1. Show that the Kähler form ω is harmonic.

2. Show that holomorphic $(n, 0)$-forms on X are harmonic, and harmonic $(n, 0)$-forms are holomorphic. (Note that this means the space of harmonic $(n, 0)$-forms on X is actually independent of the Kähler metric we choose.)
3. Show that any two cohomologous Kähler forms ω, ω' are related by $\omega = \omega' + i\partial\bar{\partial}f$ for some real function f.

Exercise 7

A closed 2-form ω on a C^∞ manifold M of dimension $2m$ is called a *symplectic structure* if ω is closed and nondegenerate at every point.

1. Show that a compact symplectic manifold has $b_{2k} \geq 1$ for $0 \leq k \leq m$. (Hint: show that ω^m is not exact.)
2. Show that Kähler manifolds carry natural symplectic structures.