

Complex Geometry: Exercise Set 4

Exercise 1

Show that the sheafification \mathcal{F}^+ of a sheaf \mathcal{F} is canonically isomorphic to \mathcal{F} itself.

Exercise 2

Fix an abelian group G . Suppose that \mathcal{F} is the constant presheaf $\mathcal{F}(U) = G$. What is the sheafification \mathcal{F}^+ ?

Exercise 3

Suppose \mathcal{F} is a sheaf of \mathcal{O}_X -modules over a complex manifold X . Show that \mathcal{F} is the sheaf of sections of a holomorphic vector bundle of rank r if and only if \mathcal{F} is locally free over \mathcal{O}_X of rank r .

Exercise 4

Suppose \mathcal{F} is a sheaf of abelian groups over M and $\phi : M \rightarrow N$ is a continuous map. Define the *direct image* $\phi_*\mathcal{F}$ by $\phi_*\mathcal{F}(U) = \mathcal{F}(\phi^{-1}(U))$.

1. Show that $\phi_*\mathcal{F}$ is a sheaf.

(We used a special case of this implicitly in lecture when we discussed the Čech resolution: there we had the inclusion maps $i_I : U_I \hookrightarrow M$, and we defined $C^k = \bigoplus_{|I|=k+1} (i_I)_*(\mathcal{F}|_{U_I})$.)

2. Suppose $M \rightarrow N$ is a covering map of degree d , and \mathcal{F} is the sheaf of sections of a holomorphic vector bundle of rank n . Show that $\phi_*\mathcal{F}$ is the sheaf of sections of a holomorphic vector bundle of rank nd . (It is probably simplest to use the equivalent characterization in terms of locally free \mathcal{O} -modules.)
3. Say $M = \mathbb{C}$, $N = \mathbb{C}$, and $\phi(z) = z^2$. What is $\phi_*(\mathcal{O})$? (There are two obvious possibilities: either $\phi_*(\mathcal{O})$ is the sheaf of sections of a rank 2 holomorphic vector bundle, or it is something more complicated because of the ramification at $z = 0$.)

Exercise 5

You may be surprised that sheaves naturally *push forward* since we have been emphasizing the point of view that a sheaf is a kind of generalization of a vector bundle, and vector bundles naturally *pull back*.

We can define the *inverse image* of a sheaf, with a bit more difficulty. Given $\phi : M \rightarrow N$ continuous, define $\phi^{-1}\mathcal{F}$ to be the sheafification of $U \mapsto \lim_{V \supset \phi^{-1}(U)} \mathcal{F}(V)$. (Note that if $i : S \rightarrow M$ is the inclusion of a closed subset, then $(i^{-1}\mathcal{F})(S)$ is what we defined in lecture to be $\mathcal{F}(S)$.)

1. Show by example that the sheafification is really necessary in this definition.
2. Show by example that if \mathcal{F} is the sheaf of sections of a holomorphic vector bundle F , $\phi^{-1}\mathcal{F}$ is generally *not* the sheaf of sections of ϕ^*F (unfortunately). Indeed, if

$\phi : X \rightarrow Y$ and \mathcal{F} is a sheaf of \mathcal{O}_Y -modules, then $\phi^{-1}\mathcal{F}$ is not even a sheaf of \mathcal{O}_X -modules. (We could say something similar about C^∞ bundles etc, replacing \mathcal{O} by the sheaf of C^∞ functions, or even more generally by any sheaf of rings.)

3. To fix this problem, when \mathcal{F} is an \mathcal{O}_Y -module, we can define

$$\phi^*\mathcal{F} = \phi^{-1}\mathcal{F} \otimes_{\phi^{-1}(\mathcal{O}_Y)} \mathcal{O}_X.$$

(This definition should be interpreted with *sheafification*, as usual for operations on sheaves.) This amounts to forcing $\phi^*\mathcal{F}$ to be a sheaf of \mathcal{O}_X -modules “by hand.” Show that if \mathcal{F} is the sheaf of sections of F then $\phi^*\mathcal{F}$ really is the sheaf of sections of ϕ^*F . (It is probably a good idea to first consider the simple case of a covering map, say 2-1.)

Exercise 6

(For those who like counterexamples.) One might have tried to define the sheafification \mathcal{F}^+ of a presheaf \mathcal{F} by taking $\mathcal{F}^+(U)$ to be the space of “discontinuous sections” $s \in \prod_{x \in U} \mathcal{F}_x$, subject to the condition that there exist a covering of U by open sets U_i with $f_i \in \mathcal{F}(U_i)$, $f_i|_{U_{ij}} = f_j|_{U_{ij}}$, and $(f_i)_x = s_x$. This doesn’t quite work if your presheaf is crazy enough: since \mathcal{F} is only a presheaf, \mathcal{F}^+ may involve coverings by sections that agree on stalks but don’t agree on intersections! Read and understand the counterexample described at

<http://mathoverflow.net/questions/31372/>

Naturally, this kind of thing won’t bother us in the rest of the course.