Hodge theory for Kähler manifolds

Def X compact Hermitian

$\Omega^\bullet_{\mathbb{C}}(X)$ has inner product $\langle \alpha, \beta \rangle_{L^2} = \int_X g_{\mathbb{C}}(\alpha, \beta)\, d\mathrm{vol}$

$$= \int_X \alpha \wedge \ast\bar{\beta}$$

Lemma X compact Hermitian: the decompositions

- $\Omega^\bullet_{\mathbb{C}}(X) = \bigoplus \Omega^k_{\mathbb{C}}(X)$,
- $\Omega^k_{\mathbb{C}}(X) = \bigoplus_{p,q} \Omega^{p,q}(X)$

are orthogonal for $\langle\, , \rangle_{L^2}$.

Pf Use $\ast : \Omega^{p,q}(X) \to \Omega^{m-q,m-p}(X)$ and $\int_X \omega = 0$ for $\omega \in \Omega^{p,q}(X)$ $p \neq m$ or $q \neq m$

Prop With respect to $\langle\, , \rangle_{L^2}$ $\partial^*, \bar{\partial}^*$ are formal adjoints to $\partial, \bar{\partial}$ resp.

Pf As for d^* above.

Lemma $\Delta_{\bar{\partial}} \alpha = 0 \iff \bar{\partial}\alpha = \bar{\partial}^* \alpha = 0$

Pf As for Δ above.

Def X Hermitian: $\mathcal{H}^k_{\bar{\partial}}(X) = \ker \Delta_{\bar{\partial}} \subset \Omega^k_{\mathbb{C}}(X)$

$\mathcal{H}^{p,q}_{\bar{\partial}}(X) = \ker \Delta_{\bar{\partial}} \subset \Omega^{p,q}_{\mathbb{C}}(X)$ (and similarly for ∂)

$\mathcal{H}^{p,q}(X) = \ker \Delta \subset \Omega^{p,q}(X)$

Prop X Hermitian: $\mathcal{H}^k_{\bar{\partial}}(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar{\partial}}(X)$, similarly for ∂

Pf Easy (just the fact that $\Delta_{\bar{\partial}}$ preserves degree)

Prop If X Kähler, $\mathcal{H}^{p,q}_{\bar{\partial}}(X) = \mathcal{H}^{p,q}(X) = \mathcal{H}^{p,q}_{\partial}(X)$

Pf From $\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta$.

Cor If X Kähler, $\mathcal{H}^k(X) \otimes \mathbb{C} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X)$

Prop X Hermitian:
$$\star : \mathcal{H}^{p,q}(X) \xrightarrow{\sim} \mathcal{H}^{n-q,n-p}(X)$$
$$\star : \mathcal{H}^{p,q}_{\bar\partial}(X) \xrightarrow{\sim} \mathcal{H}^{n-q,n-p}_{\bar\partial}(X)$$

Pf Use $\Delta\star = \star\Delta$, $\Delta_{\bar\partial}\star = \star\Delta_{\bar\partial}$.

Prop X compact Hermitian:
$$\mathcal{H}^{p,q}_{\bar\partial}(X) \times \mathcal{H}^{n-p,n-q}_{\bar\partial}(X) \longrightarrow \mathbb{C}$$
$$(\alpha, \beta) \longmapsto \int \alpha \wedge \beta$$
is <u>nondegenerate</u>. Thus $\mathcal{H}^{p,q}_{\bar\partial}(X) \simeq \mathcal{H}^{n-p,n-q}_{\bar\partial}(X)^*$ (Serre duality)

Pf $\alpha \in \mathcal{H}^{p,q}_{\bar\partial}(X) \Rightarrow \star\bar\alpha \in \mathcal{H}^{n-p,n-q}_{\bar\partial}(X)$

and if $\alpha \neq 0$ then $\int \alpha \wedge \star\bar\alpha = \|\alpha\|^2 \neq 0$ ∎

Def (Dolbeault cohomology) X complex manifold: $H^{p,q}(X) = \dfrac{\ker(\bar\partial : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X))}{\operatorname{Im}(\bar\partial : \Omega^{p,q-1}(X) \to \Omega^{p,q}(X))}$

$$h^{p,q}(M) = \dim_{\mathbb{C}} H^{p,q}(X)$$

Thm (Hodge) If X compact Kähler

Then each class in $H^{p,q}(X)$ contains a unique element of $\mathcal{H}^{p,q}(X)$

Pf Very similar to Hodge thm for Riemannian manifolds.
As in that case, main technical point is:

Lemma If X compact Kähler,
$$\Omega^{p,q}(X) = \mathcal{H}^{p,q}(X) \oplus \bar\partial\mathcal{H}^{p,q-1}(X) \oplus \bar\partial^*\mathcal{H}^{p,q+1}(X)$$
$$= \mathcal{H}^{p,q}(X) \oplus \partial\mathcal{H}^{p-1,q}(X) \oplus \partial^*\mathcal{H}^{p+1,q}(X)$$

Cor ($\partial\bar\partial$-lemma)

If X compact Kähler, $\alpha \in \Omega^{p,q}(X)$, $d\alpha = 0$,

Then TFAE:

1) α is d-exact
2) α is ∂-exact
3) α is $\bar\partial$-exact
4) α is $\partial\bar\partial$-exact

Pf $4 \Rightarrow 1,2,3$ is easy.

Hodge lemma: any of $1,2,3 \Rightarrow \alpha \perp \mathcal{H}^{p,q}(X)$.

But also $d\alpha = 0$, so Hodge lemma for ∂ says $\alpha = \partial\gamma$.

Then use Hodge lemma for $\bar\partial$ on γ to get $\gamma = \bar\partial\beta + \bar\partial^*\beta' + \beta''$ with $\beta'' \in \mathcal{H}^{p,q}(X)$

So $\alpha = \partial\bar\partial\beta + \partial\bar\partial^*\beta' = \partial\bar\partial\beta - \bar\partial^*\partial\beta'$

But $\bar\partial\alpha = 0$, so $\bar\partial\bar\partial^*\partial\beta' = 0$; thus $\langle \bar\partial\bar\partial^*\partial\beta', \partial\beta' \rangle = 0$, so $\bar\partial^*\partial\beta' = 0$,

giving finally $\alpha = \partial\bar\partial\beta$.

So any of $1,2,3 \Rightarrow 4$. ∎

And most interestingly:

Cor • X compact Kähler:

$$H^k_{dR}(X) \otimes \mathbb{C} \simeq \bigoplus_{p+q=k} H^{p,q}(X)$$

and this isomorphism is <u>independent</u> of the Kähler metric.

• $H^{p,q}(X) = \overline{H^{q,p}(X)}$

• $H^{p,q}(X) \simeq H^{n-p,n-q}(X)^*$

Pf $H^k_{dR}(X) \otimes \mathbb{C} = \mathcal{H}^k(X) \otimes \mathbb{C} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X) = \bigoplus_{p+q=k} H^{p,q}(X)$

Need to check it's indep of the Kähler metric.

Consider α, α' harmonic for 2 <u>different</u> metrics, inducing the same elt in $H^{p,q}(X)$. Want to see they induce the same elt in $H^k_{dR}(X, \mathbb{C})$.

$\alpha' - \alpha = \bar{\partial}\gamma$, and $\alpha' - \alpha$ is d-closed $\Rightarrow$

$\alpha' - \alpha = d\beta$ by $\partial\bar{\partial}$-lemma. $\blacksquare$

This decomposition gives a lot of information.

Prop 1) $h^{p,q} = h^{q,p} = h^{n-p,n-q} = h^{n-p,n-q}$

2) $b_k = \sum_{p+q=k} h^{p,q}$

3) b_k is <u>even</u> if k is odd

4) $h^{1,1} \geq 1$

Pf 1,2,3 are direct from Cor above.

For 4, use the fact that ω is harmonic.

(Actually, could have proven this more easily using $\omega^m = \text{vol}$) $\blacksquare$

Generally arrange the $h^{p,q}$ into a diamond,

$$\begin{array}{ccccc} & & h^{m,m} & & \\ & \cdot^{\cdot^{\cdot}} & & \ddots & \\ h^{m,0} & & & & h^{0,m} \\ & \ddots & h^{1,1} & \cdot^{\cdot^{\cdot}} & \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

<u>Ex</u> Let C be a surface of genus g.
Equip it w/ a $\mathbb{C}$ str and Kähler metric.
Then its Hodge diamond is

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

$$\Rightarrow \quad H^0(C,\Omega^1) = g.$$

NB, we really need <u>compactness</u> for this stuff.

<u>Ex</u> If $X = \mathbb{C}^\times$ then $b_1(X) = 1$, <u>not</u> even!

More structure:

<u>Prop</u> X cpt Kähler: $H^\bullet_{dR}(X)$ is a representation of $sl_2\mathbb{R}$, depending on the Kähler metric only via $[\omega]$.
("Hard Lefschetz")

<u>Pf</u> L, Λ map harmonic forms to harmonic forms. Thus L, H, Λ give an $sl_2\mathbb{R}$ action on $H^\bullet_{dR}(X)$.
L is cup-product with $[\omega] \Rightarrow$ depends only on $[\omega]$. Likewise H.
To see Λ also depends only on $[\omega]$ is trickier. First note that
$\ker(\Lambda \text{ on } \Omega^k) = \ker(L^{m-k+1} \text{ on } \Omega^k)$, so that $\ker(\Lambda)$ depends only on $[\omega]$.
But Λ is determined by $L, H,$ and $\ker(\Lambda)$. [e.g. $\Lambda(L\alpha) = [\Lambda, L]\alpha = H\alpha$ if $\alpha \in \ker \Lambda$]

∎

<u>Cor</u> For $p+q \le m$, $h^{p,q} \ge h^{p-1,q-1}$.

<u>Def</u> <u>Primitive cohomology</u> $H^k_p = H^k \cap Ker(\Lambda)$. Similarly $H^{p,q}_p$.