Def A seq. of sheaf homs

$$\cdots \longrightarrow \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \xrightarrow{\varphi^{i+1}} \mathcal{F}^{i+2} \longrightarrow \cdots$$

is a <u>complex</u> if $\varphi^{i+1} \circ \varphi^i = 0 \;\; \forall i$

<u>exact complex</u> if $\mathrm{Ker}(\varphi^{i+1}) = \mathrm{Im}(\varphi^i) \;\; \forall i.$

(equivalently, if the induced complex on <u>stalks</u> $\mathcal{C}_x$ is exact $\forall x$)

Ex $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathcal{O}^\times \xrightarrow{f^2} \mathcal{O}^\times \to 0, \qquad 0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to 0.$

$$0 \longrightarrow \mathbb{R} \xrightarrow{i} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \longrightarrow 0$$

Prop If $0 \to \mathcal{F} \xrightarrow{\psi} \mathcal{G} \xrightarrow{\varphi} \mathcal{H}$ is exact

then $0 \to \mathcal{F}(U) \xrightarrow{\psi_U} \mathcal{G}(U) \xrightarrow{\varphi_U} \mathcal{H}(U)$ is exact

at $\mathcal{F}(U)$ and $\mathcal{G}(U)$.

Pf At $\mathcal{F}(U)$: say $\alpha \in \mathcal{F}(U)$ maps to zero, then each α_x maps to zero, then each $\alpha_x = 0$, then $\alpha = 0$.

At $\mathcal{G}(U)$: say $\beta \in \mathcal{G}(U)$ has $\psi_U(\beta) = 0$.
Exactness on <u>stalks</u> $\Rightarrow \exists$ covering by $\{U_i\}$ with $\alpha_i \in \mathcal{F}(U_i)$, $\beta|_{U_i} = \psi_{U_i}(\alpha_i)$. But $\psi_{U_{ij}}(\alpha_i - \alpha_j) = 0 \Rightarrow \alpha_i = \alpha_j$ on U_{ij}
which means the α_i glue together into $\alpha \in \mathcal{F}(U)$ ■

Def $\mathcal{F}$ a sheaf: a <u>resolution</u> of $\mathcal{F}$ is an exact seq of sheaves

$$0 \longrightarrow \mathcal{F} \xrightarrow{i} R^0 \longrightarrow R^1 \longrightarrow R^2 \longrightarrow R^3 \longrightarrow \cdots$$

Ex <u>De Rham resolution</u> (on C^∞ M): $\quad 0 \longrightarrow \underline{\mathbb{R}} \xrightarrow{i} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \longrightarrow \cdots$

<u>Dolbeault resolution</u> (on cplx X): $\quad 0 \to \Omega^p \xrightarrow{i} \Omega^{p,0} \xrightarrow{\bar\partial} \Omega^{p,1} \xrightarrow{\bar\partial} \Omega^{p,2} \to \cdots$

or, if E hol v.b., $\quad 0 \to E \xrightarrow{i} \Omega^{0,0}(E) \xrightarrow{\bar\partial_E} \Omega^{0,1}(E) \xrightarrow{\bar\partial_E} \Omega^{0,2}(E) \to \cdots$

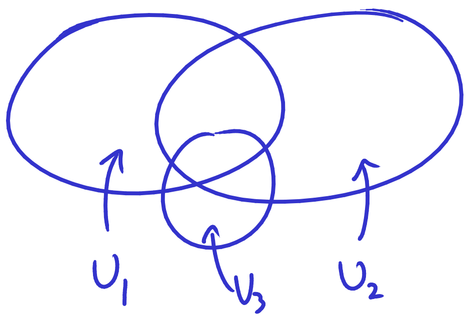
Čech resolution: Fix a countable covering $\{U_i\}$. Define $U_I = \bigcap_{i\in I} U_i$.

Then define $\mathcal{F}_I(U) = \mathcal{F}(U \cap U_I)$ [it's a sheaf]

$$\mathcal{F}^k = \bigoplus_{|I|=k+1} \mathcal{F}_I$$

$d: \mathcal{F}^k \to \mathcal{F}^{k+1}$ by $\displaystyle (d\sigma)_{j_0,\dots,j_{k+1}} = \sum_i (-1)^i \sigma_{j_0,\dots,\hat{j_i},\dots,j_{k+1}}\Big|_{U\cap U_{j_0\dots j_{k+1}}}$

[Ex



$\sigma \in \mathcal{F}^0$ is a triple $(\sigma_1,\sigma_2,\sigma_3) \in \mathcal{F}(U_1)\times\mathcal{F}(U_2)\times\mathcal{F}(U_3)$

$d\sigma \in \mathcal{F}^1$ is $(\sigma_1-\sigma_2,\ \sigma_2-\sigma_3,\ \sigma_1-\sigma_3) \in \mathcal{F}(U_{12})\times\cdots$

$d^2\sigma$ is $((\sigma_1-\sigma_2)+(\sigma_2-\sigma_3)-(\sigma_3-\sigma_1)) = 0 \in \mathcal{F}(U_{123})$]

Then $0 \to \mathcal{F} \xrightarrow{i} \mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1 \xrightarrow{d} \cdots$ is a resolution of $\mathcal{F}$.

(Depends on the covering!)

Singular cochains: M a topological manifold.

$U \mapsto$ gp. of singular cochains in U w/ coeff. in G is a presheaf.

Sheafify to get a sheaf $S^p(G)$.

Then get a resolution $0 \to \underline{G} \to S^0(G) \xrightarrow{\delta} S^1(G) \xrightarrow{\delta} \cdots$

Any of these resolutions can be used to compute cohomology, as we'll see...