Calculating sheaf cohomology

Given an open cover $\mathcal{U} = \{U_i\}$ of M we defined the <u>Čech resolution</u> $C^\cdot(\mathcal{U})$ of any sheaf $\mathcal{F}$ over M. Recall $U_I = \bigcap_{i \in I} U_i$.

<u>Prop</u> Suppose $H^p(U_I, \mathcal{F}) = 0$ $\forall p > 0, I$. [Leray]
Then $H^p(M, \mathcal{F}) = H^p(C^\cdot(\mathcal{U})(\mathcal{F}))$

<u>Pf</u> By "dimension-shifting" — see
http://www.math.umn.edu/~garrett/m/algebra/cech.pdf page 9
(replace I by Godement's flabby sheaf $C^0(S)$)

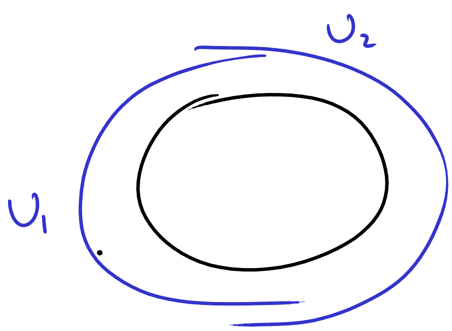
[In fact, one can also show on paracompact M that you can <u>always</u> pass to a fine enough cover. (Godement) Could <u>define</u> $H^p(M, \mathcal{F})$ this way.]

How to produce good covers?

<u>Lemma</u> U contractible $\Rightarrow H^p(U, \underline{G}) = 0$ $p > 0$

<u>Pf</u> Use resolution by singular cochains. (Is there a more direct way?)

<u>Ex</u> $M = S^1$



Calculate $H^\cdot(M, \underline{\mathbb{Z}})$:

$$0 \to C^0(\mathcal{U}, \underline{\mathbb{Z}}) \xrightarrow{d} C^1(\mathcal{U}, \underline{\mathbb{Z}}) \to 0$$

$$C^0(\mathcal{U}, \underline{\mathbb{Z}}) = \underline{\mathbb{Z}}(U_1) \oplus \underline{\mathbb{Z}}(U_2), \qquad C^1(\mathcal{U}, \underline{\mathbb{Z}}) = \underline{\mathbb{Z}}(U_1 \cap U_2)$$

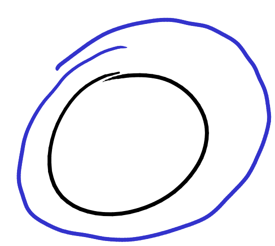
$$(a, b) \longmapsto (a-b, a-b)$$

$$H^0(M, \underline{\mathbb{Z}}) = \mathrm{Ker}(d) \simeq \mathbb{Z}$$

$$H^1(M, \underline{\mathbb{Z}}) = \frac{C^1(\mathcal{U}, \underline{\mathbb{Z}})}{\mathrm{Im}(d)} \simeq \mathbb{Z}$$

$$H^p(M, \underline{\mathbb{Z}}) = 0 \text{ for } p > 1$$

[Remark: Can also use just <u>one</u> "open set" (<u>etale cover</u>)


]

<u>Ex</u> Similarly $H^p(\mathbb{C}^\times, \underline{\mathbb{Z}/2\mathbb{Z}}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } p = 0, 1 \\ 0 & \text{for } p > 1 \end{cases}$

How about analytic sheaves?

Def A *polydisc* in $\mathbb{C}^m$ is $\{\vec{z} : |z_i| \leq \varepsilon_i\}$ where each $\varepsilon_i \in \mathbb{R} \cup \{\infty\}$.

Ex $X = \text{polydisc} \Rightarrow H^p(X, \mathcal{O}) = 0 \ \forall p > 0$

(Because $H^p(X,\mathcal{O}) \simeq H^{0,p}(X)$, which vanishes by $\bar\partial$-Poincaré lemma)

Ex X polydisc $\Rightarrow H^p(X, \mathcal{O}^\times) = 0 \quad p > 0$

(Use $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 0$)

This admits a massive, important generalization:

Thm ("Cartan's Theorem B") If X is a Stein manifold ($\simeq$ closed $\mathbb{C}$ submfld of $\mathbb{C}^n$) and $\mathcal{F}$ a coherent sheaf then $H^p(X, \mathcal{F}) = 0 \ \forall p > 0$.

Ex $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathcal{O}^\times \xrightarrow{\varphi} \mathcal{O}^\times \to 0 \qquad M = \mathbb{C}^\times$

$\varphi(f) = f^2$

$$H^0(\mathbb{C}^\times, \mathbb{Z}/2\mathbb{Z}) \to H^0(\mathbb{C}^\times, \mathcal{O}^\times) \to H^0(\mathbb{C}^\times, \mathcal{O}^\times) \xrightarrow{\delta} H^1(\mathbb{C}^\times, \mathbb{Z}/2\mathbb{Z})$$

$H^0(\mathbb{C}^\times, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, $\quad H^1(\mathbb{C}^\times, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

Ex What is δ *concretely*?