Double complexes

We showed that given 2 acyclic resolutions of a sheaf $\mathcal{F}$

$$0 \to \mathcal{F} \to \mathcal{A}^\bullet$$
$$0 \to \mathcal{F} \to \mathcal{B}^\bullet$$

we get

$$H^\bullet(M,\mathcal{F}) \simeq H^\bullet(\mathcal{A}^\bullet(M))$$
$$\simeq H^\bullet(\mathcal{B}^\bullet(M))$$

But how to compare them concretely?

One way: build a double complex of sheaves,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{A}^0 \arrow[r, "d"] \arrow[d, "\iota_A"] & \mathcal{A}^1 \arrow[r, "d"] \arrow[d, "\iota_A"] & \mathcal{A}^2 \arrow[r] \arrow[d, "\iota_A"] & \cdots \\
0 \arrow[r] & \mathcal{B}^0 \arrow[r, "\iota_B"] \arrow[d, "\delta"'] & \mathcal{C}^{0,0} \arrow[r, "d"] \arrow[d, "\delta"] & \mathcal{C}^{0,1} \arrow[r, "d"] \arrow[d, "\delta"] & \mathcal{C}^{0,2} \arrow[r] \arrow[d, "\delta"] & \cdots \\
0 \arrow[r] & \mathcal{B}^1 \arrow[r, "\iota_B"] \arrow[d, "\delta"'] & \mathcal{C}^{1,0} \arrow[r, "d"] \arrow[d, "\delta"] & \mathcal{C}^{1,1} \arrow[r, "d"] \arrow[d, "\delta"] & \ddots & \\
0 \arrow[r] & \mathcal{B}^2 \arrow[r, "\iota_B"] \arrow[d] & \mathcal{C}^{2,0} \arrow[r, "d"] \arrow[d] & \ddots & & \\
 & \vdots & \vdots & & &
\end{tikzcd}$$

$d^2 = 0$

$\delta^2 = 0$

$d\delta = \delta d$ (commuting diag.)

Associated single complex: $\mathcal{C}^k = \bigoplus_{p+q=k} \mathcal{C}^{p,q}$ with differential $D = d + (-1)^p \delta$

The inclusions ι_A, ι_B give natural maps
$i_A : H^\bullet(\mathcal{A}^\bullet(M)) \to H^\bullet(\mathcal{C}^\bullet(M))$
$i_B : H^\bullet(\mathcal{B}^\bullet(M)) \to H^\bullet(\mathcal{C}^\bullet(M))$

Prop • If all rows are acyclic resolutions, i_B is $\simeq$.

• If all columns " " " i_A is $\simeq$.

Pf Voisin p. 188 (Lemma 8.5)

Best possible case: all rows and cols acyclic.

In that case, can use $\mathcal{C}^\cdot$ to construct concrete iso. $H^\cdot(A^\cdot(M)) \simeq H^\cdot(\mathcal{B}^\cdot(M))$.

(And, claim: it's the same iso. that comes from identifying both with $H^\cdot(M,\mathcal{F})$. Why?)

For example: given $b^2 \in \mathcal{B}^2(M)$ with $\delta b^2 = 0 \rightsquigarrow [b^2] \in H^2(\mathcal{B}^\cdot(M))$

$\iota_B(b) = c^{2,0} \in \mathcal{C}^{2,0}$ with $\delta c^{2,0} = 0$, $dc^{2,0} = 0 \rightsquigarrow [c^{2,0}] \in H^2(\mathcal{C}^\cdot(M))$

so $\exists\, c^{1,0} \in \mathcal{C}^{1,0}$ with $\delta c^{1,0} = c^{2,0}$ (cols. acyclic)

then $dc^{1,0} = c^{1,1} \in \mathcal{C}^{1,1}$ with $dc^{1,1} = 0$, $\delta c^{1,1} = dc^{2,0} = 0$ $\rightsquigarrow [c^{1,1}] \in H^2(\mathcal{C}^\cdot(M))$

and $Dc^{1,0} = c^{1,1} - c^{2,0} \Rightarrow [c^{2,0}] = [c^{1,1}]$

Continuing: $\exists\, c^{0,1} \in \mathcal{C}^{0,1}$ with $\delta c^{0,1} = c^{1,1}$ (cols acyclic)

then $dc^{0,1} = c^{0,2} \in \mathcal{C}^{0,2}$ with $dc^{0,2} = 0$, $\delta c^{0,2} = dc^{1,1} = 0$ $\rightsquigarrow [c^{0,2}] \in H^2(\mathcal{C}^\cdot(M))$

and $Dc^{0,1} = c^{1,1} + c^{0,2} \Rightarrow [c^{0,2}] = -[c^{1,1}]$

Finally, $\exists\, a^2 \in A^2$ with $\iota_A(a^2) = c^{0,2}$

Altogether, $i_B([b^2]) = [c^{2,0}] = [c^{1,1}] = -[c^{0,2}] = -i_A([a^2])$

This "stair-step" procedure gives the desired explicit way of passing from a representative $[b^2]$ in one realiz^n of cohomology to its counterpart $[-a^2]$ in another realiz^n. (We use it later in the course)

Of course, this depends on being able to produce the needed interpolating double complex!

One impt example arises when one of the two resolutions, say $A^\bullet$, is Čech.

Then there's a natural way to build double complex: just take Čech resolutions of all the sheaves in the resolution $\mathcal{B}^\bullet$.

i.e. $\mathcal{C}^{p,q} = C^p(\mathcal{B}^q)$

Čech