Jacobians of compact Kähler manifolds

Recall $\mathrm{Jac}(X) = \mathrm{Ker}\left(\mathrm{Pic}(X) \xrightarrow{c_1} H^2(X,\mathbb{Z})\right)$.

We noted earlier that this is $H^1(X,\mathcal{O}) / H^1(X,\mathbb{Z})$ i.e. $H^{0,1}(X) / H^1(X,\mathbb{Z})$.

Prop If X compact Kähler, then $\mathrm{Jac}(X)$ is a complex torus of real dimension $= b_1(X)$.

Pf As a real v.s., we know $H^{0,1}(X) \simeq H^1(X,\mathbb{R})$

$$\left[\begin{array}{c} H^1(X,\mathbb{C}) \simeq H^{0,1}(X) \oplus H^{1,0}(X) \\ V_{\mathbb{C}} \simeq V \oplus \bar{V} \end{array}\right]$$

and the embedding is the standard one; so $\mathrm{Jac}(X) = \dfrac{H^1(X,\mathbb{R})}{H^1(X,\mathbb{Z})}$ as a real manifold. ∎

Rk
- Holomorphic objects for some reason are parameterized by a complex m.s.!
- This result says that any cpt Kähler mfd has a natural complex torus attached. As a C^∞ mfd it just detects $b_1(X)$, but its complex structure has more information. For X of genus 1, we noticed this a while ago: $\mathrm{Jac}(X_\tau) \simeq X_\tau$, so could reconstruct X from its Jacobian.

  Higher genus –

  Torelli thm: knowing $\mathrm{Jac}(X)$ as a complex mfd plus a bit of extra discrete data ("principal polarization") is enough to reconstruct X! Schottky problem: which tori arise as $\mathrm{Jac}(X)$ for some X? For $g \geq 4$, the answer is not "all of them"...

  Families of curves and the corresponding families of Jacobians occur very often in nature ($\mathcal{N}=2$ SUSY, integrable systems...)

- There are also higher-dimensional analogues ("intermediate Jacobians"): $\forall k$ odd, complex tori

$$\frac{H^k(X,\mathbb{R})}{H^k(X,\mathbb{Z})} \simeq \frac{H^{k,0}\oplus H^{k-1,1}\oplus\cdots\oplus H^{\lceil \frac{k}{2}\rceil,\lfloor\frac{k}{2}\rfloor}(X)}{H^k(X,\mathbb{Z})} \quad \text{[Griffiths]}$$

or

$$\simeq \frac{H^{k,0}\oplus H^{k-2,2}\oplus H^{k-4,4}\oplus\cdots\oplus H^{1,k-1}}{H^k(X,\mathbb{Z})} \quad \text{[Weil]}$$

$k=1$ gives Jacobian, $k=m-1$ gives "Albanese torus" naturally dual to it. $k=3$ impt for MS.

How should we really think about $\mathrm{Jac}(X) \simeq$ real torus?

Says to any hol. line bundle we may associate some "angles" lying in $\frac{H^1(X,\mathbb{R})}{H^1(X,\mathbb{Z})}$

i.e. given a line bundle $\mathcal{L}$ and a cycle $\gamma\in H_1(X,\mathbb{Z})$ we should get a corresponding element in $\mathbb{R}/\mathbb{Z} = U(1)$.

Natural expectation: this is the <u>holonomy</u> of some flat connection.

ie, there should be some kind of corresp. between <u>hol. structures</u> on top. trivial bundles and <u>flat connections</u>. Let's explore that next...