# Kodaira vanishing

**Def** $X$ compact complex.
A line bundle $L$ is called <u>positive</u> if $c_1(L) \in H^2(X,\mathbb{R})$ can be represented by a closed +ve real $(1,1)$ form.

**Ex** On $\mathbb{CP}^n$, $\mathcal{O}(1)$ is positive (we've produced $\nabla$ s.t. $\frac{i}{2\pi}F_\nabla$ is positive before)

**Thm** If $X$ compact Kähler, $\dim_{\mathbb{C}} X = n$, and $L$ positive, then

$$H^q(X, \Omega^p \otimes L) = 0 \quad \text{for } p+q > n \qquad \text{(Kodaira-Nakano vanishing)}$$

Prove this in several steps. First, recall $[\Lambda, \bar\partial] = -i\partial^*$ where $\partial^* = -\star \circ \bar\partial \circ \star$
$= -\bar\star \circ \partial \circ \bar\star$.

Vector bundle analogue: $(\nabla^{1,0})^* = -\bar\star_{E^*} \circ \nabla^{1,0}_{E^*} \circ \bar\star_E$ [Exercise].

**Lemma** $X$ Kähler, $E$ Hermitian hol, $\nabla$ Chern connection:

$$[\Lambda, \bar\partial_E] = -i(\nabla^{1,0})^* \quad \text{acting on } \Omega^{p,q}(E)$$

**Pf** Work in orthonormal trivialization: $\bar\partial_E = \bar\partial + A^{0,1}$, $(\nabla^{1,0})^* = \partial^* - (A^{1,0})^*$

$$[\Lambda, \bar\partial_E] + i(\nabla^{1,0})^* = \underbrace{\cancel{[\Lambda, \bar\partial] + i\partial^*}}_{\text{by Kähler id.}}{}^{0} + [\Lambda, A^{0,1}] - i(A^{1,0})^*$$

And at any point we can always choose our triv so that $A=0$, so last 2 terms vanish.

(Or: more directly, $[\Lambda, A^{0,1}] = [\omega \lrcorner \cdot, A^{0,1} \wedge \cdot] = \iota_{A^{0,1}} \omega \lrcorner \cdot = i A^{1,0} \lrcorner \cdot$) ∎

**Lemma** $X$ compact complex, $E$ Hermitian hol, $\nabla$ Chern, $\alpha \in \mathcal{H}^{p,q}(X,E)$:

i) $i(F_\nabla \Lambda(\alpha), \alpha)_{L^2} \leq 0$

ii) $i(\Lambda F_\nabla(\alpha), \alpha)_{L^2} \geq 0$

**Pf** i) $i(F_\nabla \Lambda(\alpha),\alpha) = i(\nabla^{1,0}\bar\partial_E \Lambda(\alpha),\alpha) + \cancel{i(\bar\partial_E \nabla^{1,0}\Lambda(\alpha),\alpha)}$ — 0 since $\alpha$ harmonic

$$= i(\bar\partial_E \Lambda(\alpha), (\nabla^{1,0})^* \alpha)$$
$$= (\bar\partial_E \Lambda(\alpha), [\Lambda,\bar\partial_E]\alpha)$$
$$= -(\bar\partial_E \Lambda(\alpha), \bar\partial_E \Lambda(\alpha)) \leq 0.$$

ii) similar.

**Pf of Thm**

Pick a Herm. metric $h$ in $L$. Then $L$ positive $\Rightarrow \frac{i}{2\pi}F_\nabla + d\alpha$ is a positive $(1,1)$-form for some $\alpha$. Then $d\alpha = \frac{i}{2\pi}\partial\bar\partial\varphi$ for some real $f^n$ $\varphi$ ($\partial\bar\partial$-lemma). Replace $h \to e^\varphi h$. Then $\frac{i}{2\pi}F_\nabla$ is positive; use it as a Kähler form.

Then $$0 \leq i([\Lambda, F_\nabla](\alpha),\alpha) = 2\pi([\Lambda,L]\alpha,\alpha)$$
$$= 2\pi(n-(p+q))\|\alpha\|^2_{L^2}$$ [by Kähler identity $[\Lambda,L]=H$]

So $n \geq p+q$ or $\alpha = 0$ ∎

**Ex** For $X = \mathbb{CP}^n$:

Thm $\Rightarrow H^q(X, K\otimes\mathcal{O}(m)) = 0 \quad q>0, m>0$

i.e. $H^q(X,\mathcal{O}(m)) = 0 \quad q>0, m>-n-1$

Using Serre duality $h^q(X,\mathcal{O}(m)) = h^{n-q}(X,\mathcal{O}(-n-1-m))$ and <u>knowing</u> $h^0(X,\mathcal{O}(m))$, can compute all $h^q(X,\mathcal{O}(m))$:

| $q \backslash m$ | $\cdots$ | $-n-2$ | $-n-1$ | $-n$ | $\cdots$ | $-1$ | $0$ | $1$ | $2$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $0$ | $\cdots$ | 0 | 0 | 0 | $\cdots$ | 0 | 1 | $n+1$ | $\binom{n+2}{2}$ | $\cdots$ |
| $1$ | $\cdots$ | 0 | 0 | 0 | $\cdots$ | 0 | 0 | 0 | 0 | $\cdots$ |
| $\vdots$ | | | | | | | | | | |
| $n-1$ | $\cdots$ | 0 | 0 | 0 | $\cdots$ | 0 | 0 | 0 | 0 | $\cdots$ |
| $n$ | $\cdots$ | $n+1$ | 1 | 0 | $\cdots$ | 0 | 0 | 0 | 0 | $\cdots$ |

Similarly:

**Thm** $L$ positive, $X$ compact Kähler, $E$ hol $\Rightarrow$ $\exists\, m_0$ s.t.

$$H^q(X, E \otimes L^m) = 0 \quad \text{for } m > m_0.$$

**Pf** See Huybrechts.

("Twisting by a big enough positive line bundle kills all the higher cohomology."
Like a thm of Serre in algebraic setting.)

By Serre duality, this also $\Rightarrow$ twisting by a big enough negative line bundle kills all the global sections.