Riemannian Geometry: Exercise Set 3

Exercise 1

Define a *conformal map* ϕ between Riemannian manifolds (M, g) and (M', g') to be one for which $\phi^* g' = fg$, for some nowhere-vanishing $f \in C^\infty(M)$.

1. Show that ϕ is conformal if and only if it preserves (the cosines of) angles between tangent vectors.
2. Suppose M, M' are open subsets of the complex plane with its usual flat metric. Show that $\phi : M \to M'$ is conformal if and only if it is either holomorphic or antiholomorphic and has no critical points. What is the rescaling factor f? (Hint: it is convenient to write the metric as $g = \frac{1}{2} \mathrm{d}z\, \mathrm{d}\bar{z}$; but if you do this you should be careful to understand exactly what it means!)
3. (For fun.) Give a direct "geometric" argument — no grubby computations, just pictures — that stereographic projection is an angle-preserving (hence conformal) map from a patch of S^n to $\mathbb{R}^n$.

Exercise 2

1. Do Exercise 3.11 of Lee (proof that the Poincare half-space model of hyperbolic space is isometric to the Poincare ball model — the notation is defined on page 38).

Exercise 3

Suppose M is a smooth manifold with vector bundles E, E' with connections ∇, ∇'.

1. Fix bases for E and E'. Write the connection coefficients for the canonical induced connections on $E \oplus E'$, $E \otimes E'$, E^* and $\mathrm{End}(E)$ (with respect to the induced bases), in terms of the connection coefficients for ∇ and ∇'.

Exercise 4

Suppose M is a smooth manifold with a vector bundle E.

1. Suppose E has connections ∇, ∇' with $\nabla = \nabla' + A$, where $A \in \mathcal{E}(\mathrm{End}\, E \otimes T^*M)$. For any $\omega \in \mathcal{E}(E \otimes T^*M)$, show that $\mathrm{d}_\nabla \omega = \mathrm{d}_{\nabla'} \omega + A \wedge \omega$.
2. Show the "Leibniz rule": for any $s \in \mathcal{E}(E)$, $\mathrm{d}_\nabla(As) = \mathrm{d}_\nabla(A)s - A \wedge \nabla s$. (Note the tricky minus sign!)

Exercise 5

Suppose M is a smooth manifold with a vector bundle E and connection ∇.

1. Verify that if $\gamma : [0,1] \to M$ and $\tilde{\gamma}(t) = \gamma(1-t)$ then $P_{\gamma,\nabla} = P^{-1}_{\tilde{\gamma},\nabla}$.
2. Suppose $s \in \mathcal{E}(E)$ and $X \in T_pM$. Show that $\nabla_X s$ depends only on the restriction of s to a curve passing through p whose tangent vector at p is X. (In other words: fix some such curve and consider two sections s, s' which agree along the curve; then show $\nabla_X s = \nabla_X s'$.)