Lecture 2

1 Sep 2015

Composition of functions

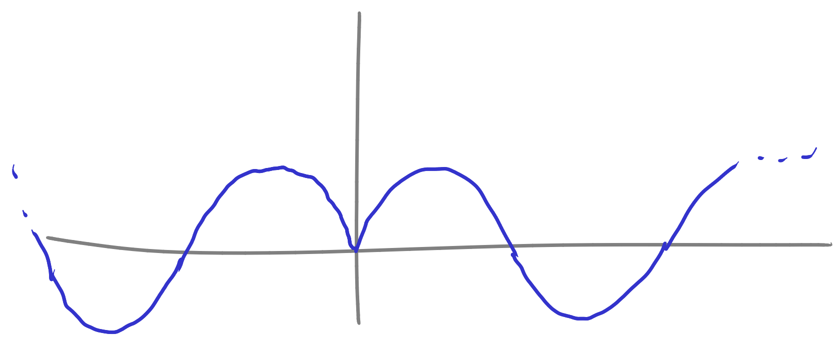
If f, g are functions, with $\mathrm{range}(g) \subset \mathrm{domain}(f)$, then we can define a <u>new</u> function $f \circ g$ by the formula $f \circ g(x) = f(g(x))$

Ex $f(x) = x^2 + 1$
$g(x) = 6x$

then $f(g(x)) = f(6x) = (6x)^2 + 1 = 36x^2 + 1$

Ex $f(x) = \sin x$
$g(x) = |x|$

$(f \circ g)(x) = f(|x|) = \sin |x|$



even function: $\sin |x| = \sin |-x|$

Inverse functions

If f is a function, its <u>inverse</u> is a function f^{-1} such that $(f^{-1} \circ f)(x) = x$

$\mathrm{domain}(f^{-1}) = \mathrm{range}(f)$
$\mathrm{range}(f^{-1}) = \mathrm{domain}(f)$

$x \rightarrow \boxed{f} \rightarrow f(x) \rightarrow \boxed{f^{-1}} \rightarrow x$

x	$f(x)$
1	7
2	11
3	18

x	$f^{-1}(x)$
7	1
11	2
18	3

Ex If $f(x) = 13x$ then $f^{-1}(x) = \frac{x}{13}$

(Because $f^{-1}(f(x)) = f^{-1}(13x) = \frac{13x}{13} = x$)

Ex If $f(x) = x+3$ then $f^{-1}(x) = x-3$

Some functions f don't have inverses:

e.g.

x	$f(x)$
1	8
2	8
3	2

x	$f^{-1}(x)$
8	1
8	2
2	3

no such function f^{-1}!

We say f is a 1-1 function if

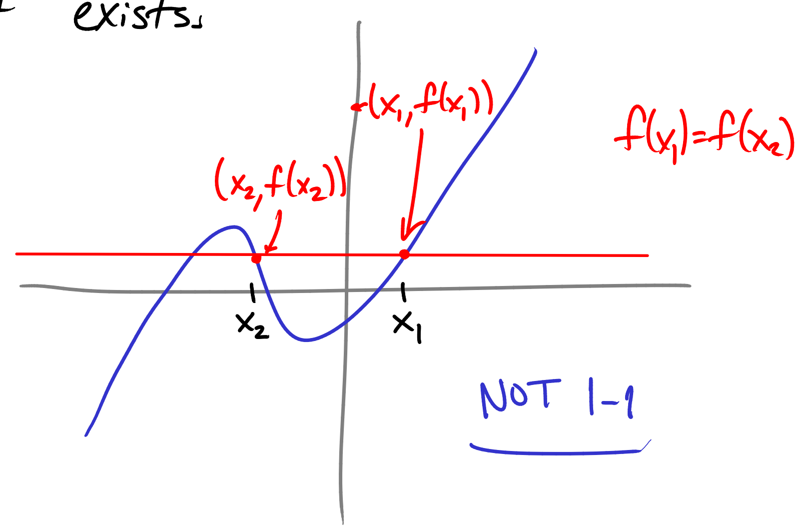
$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2$$

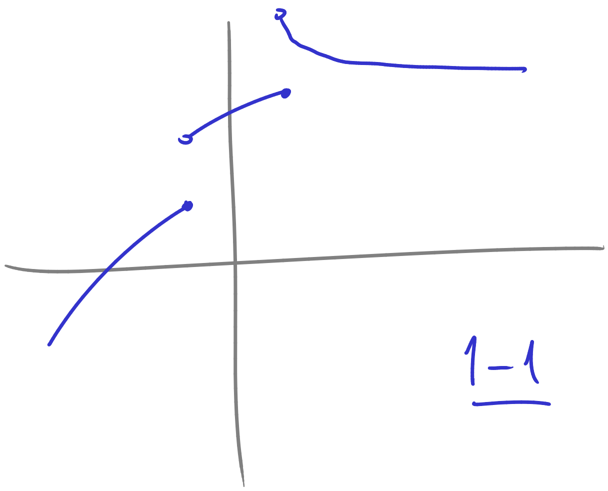
If f is a 1-1 function, then f^{-1} exists.

How to see whether f is 1-1?

Horizontal line test:

f is 1-1 just if there is no horizontal line which hits the graph of f in more than 1 point.

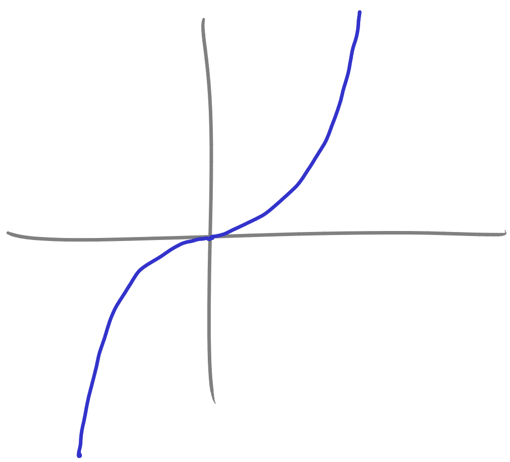




1-1

Ex $f(x) = x^3$

is 1-1.



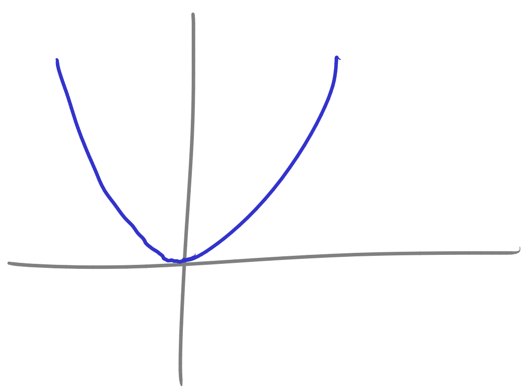
So, it has an inverse:

$f^{-1}(x) = \sqrt[3]{x}$.

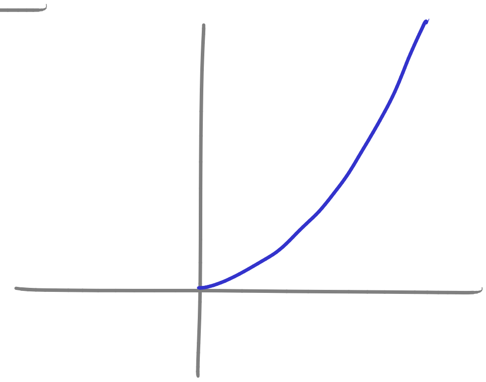
(NB, this makes sense for all real x, not only $x \geq 0$)

Ex $f(x) = x^2$

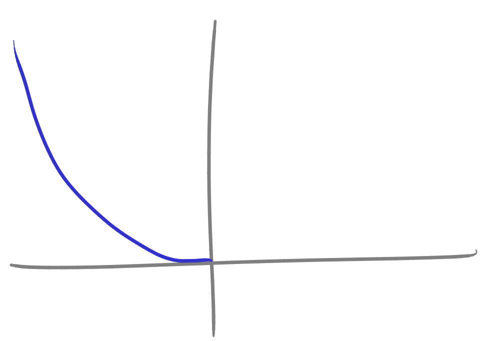
is not 1-1, if we take its domain to be $(-\infty, \infty)$.

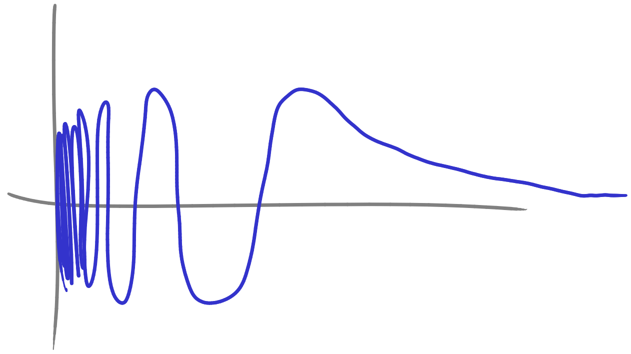


But, if we take smaller domain $[0, \infty)$ then it's 1-1, has inverse $f^{-1}(x) = \sqrt{x}$.



If we take domain $(-\infty, 0]$ then it's 1-1, has inverse $f^{-1}(x) = -\sqrt{x}$.





$f(x) = \sin\left(\frac{1}{x}\right)$

<u>Ex</u> $f(x) = x^3 - 1$ 1-1 ✓

To find the inverse: we want to <u>solve for x</u>, given $f(x)$.

Call $f(x)$ "y" for convenience.

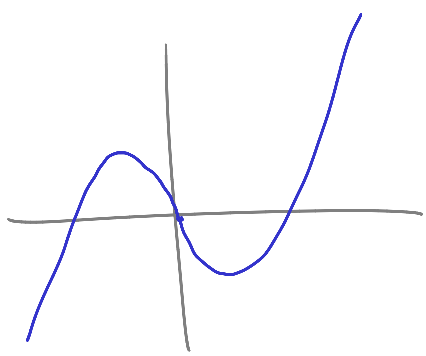
$$y = x^3 - 1$$
$$y + 1 = x^3$$
$$x = \sqrt[3]{y+1}$$

Thus $f^{-1}(y) = \sqrt[3]{y+1}$.

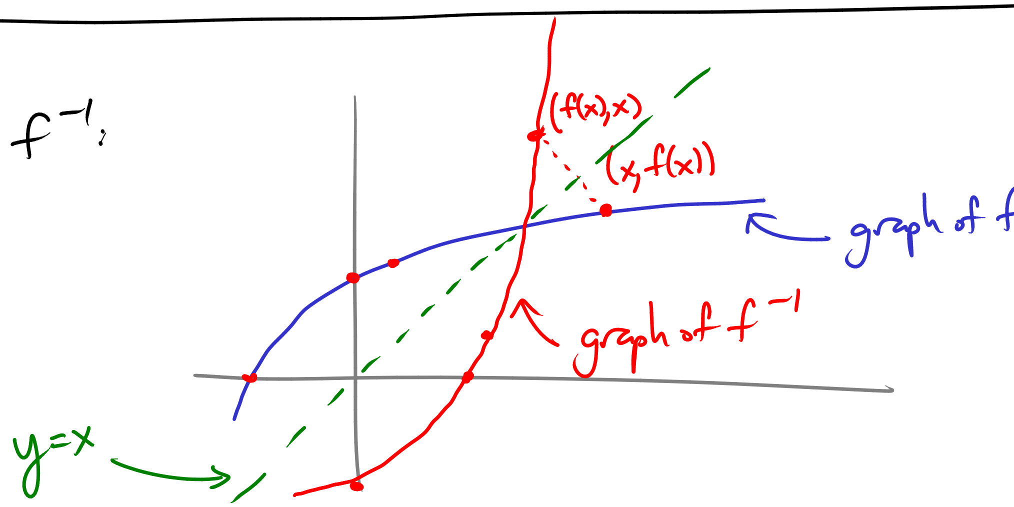
(or: $f^{-1}(x) = \sqrt[3]{x+1}$.)

<u>Ex</u> $f(x) = x^3 - x$

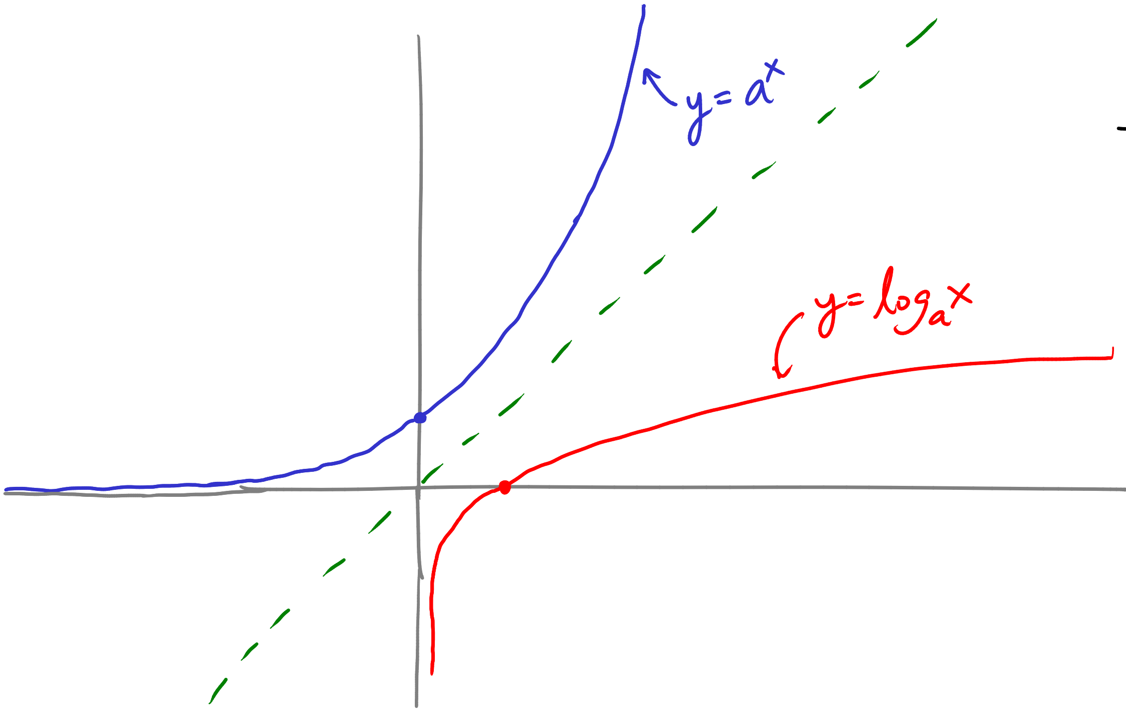
<u>not</u> 1-1, (even though $f(x) = x^3$ is 1-1)



A <u>picture</u> of f^{-1}:



Ex $f(x) = a^x$ is 1:1 if $a \neq 1$. We call its inverse $f^{-1}(x) = \log_a x$.



x	10^x
0	1
1	10
2	100
3	1000
−1	0.1
−2	0.01
−3	0.001

x	$\log_{10} x$
1	0
10	1
100	2
1000	3
.1	−1
0.01	−2
0.001	−3

Laws of logarithms

① $\log_a(xy) = \log_a x + \log_a y$

② $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

③ $\log_a(x^r) = r \log_a x$

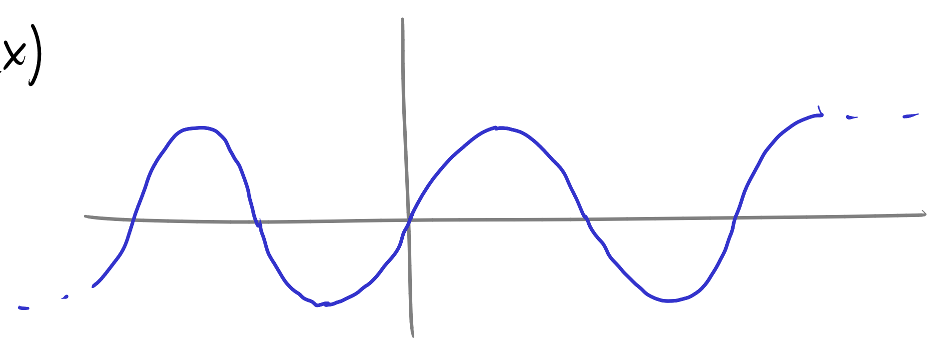
(Why? e.g. ① comes from $a^x a^y = a^{x+y}$)

Ex $\log_3 63 - \log_3 7 = \log_3 \frac{63}{7} = \log_3 9 = 2$ (because $3^2 = 9$)

Ex $f(x) = \sin(x)$

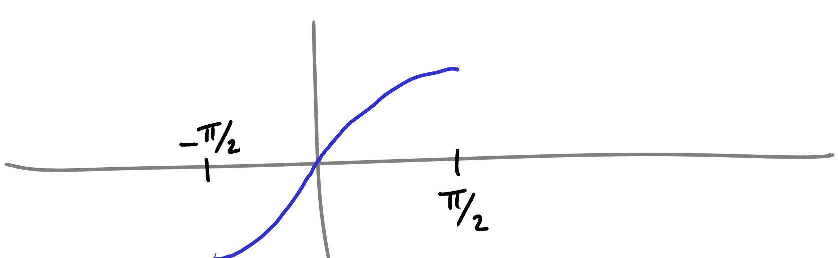
not 1-1

on $(-\infty, \infty)$



but is 1-1

on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

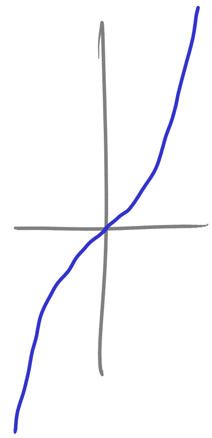


Let $\sin^{-1}$ mean the inverse of $\sin$ <u>on the domain</u> $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

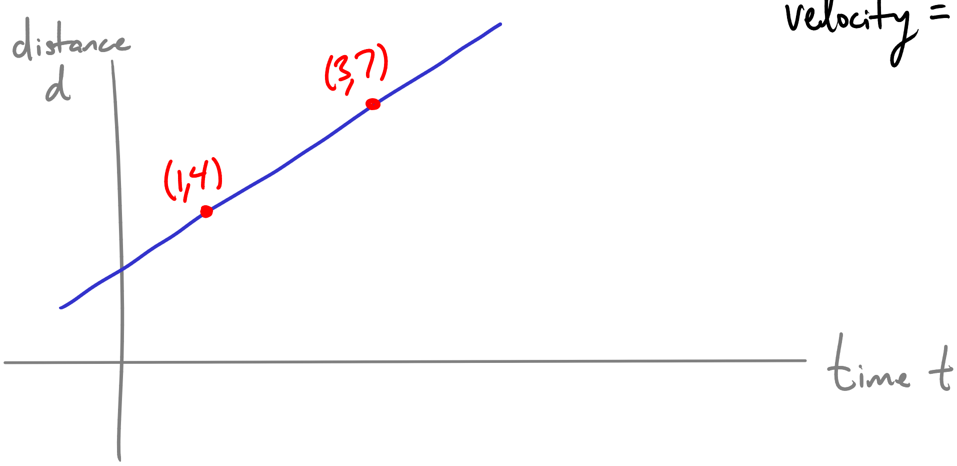
So "$\theta = \sin^{-1} x$" means "$\sin\theta = x$ <u>and</u> $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$".

Similarly for $\cos x$, using domain $[0, \pi]$.

for $\tan x$, " " $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



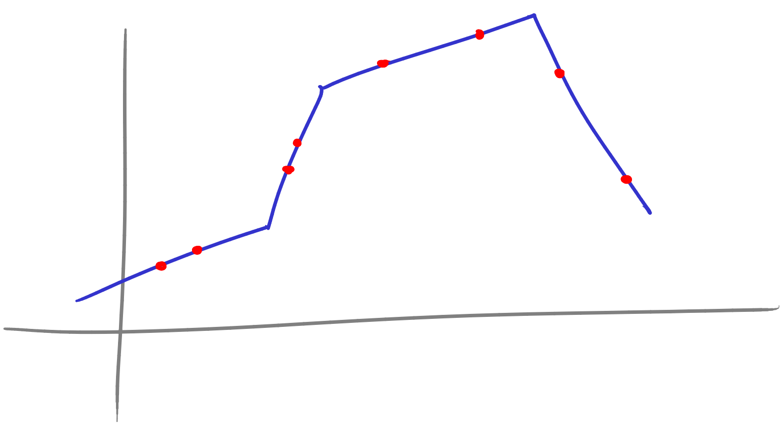
Velocity



$$\text{velocity} = \frac{7-4 \text{ miles}}{3-1 \text{ minutes}} = \frac{3 \text{ miles}}{2 \text{ min}}$$

$$= 1.5 \ \frac{\text{miles}}{\text{min}}$$

$$= 90 \text{ mph}$$



compute slope of each subinterval separately to get speed on that subinterval