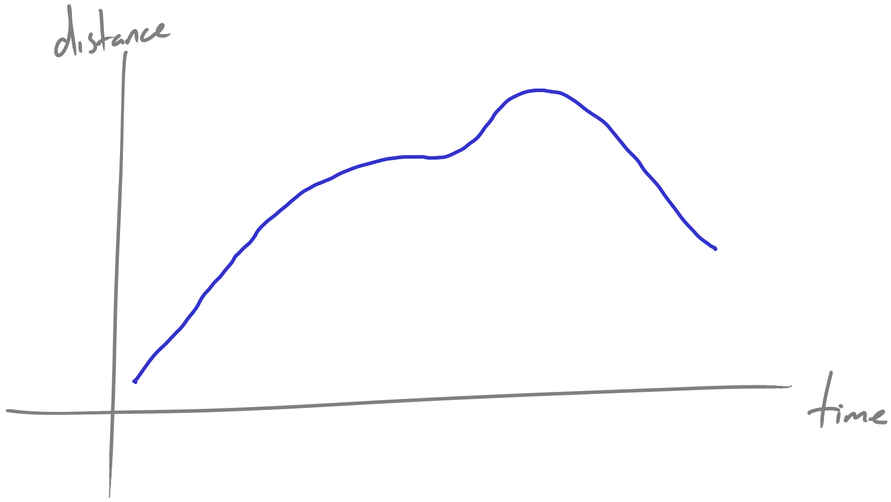
Lecture 3

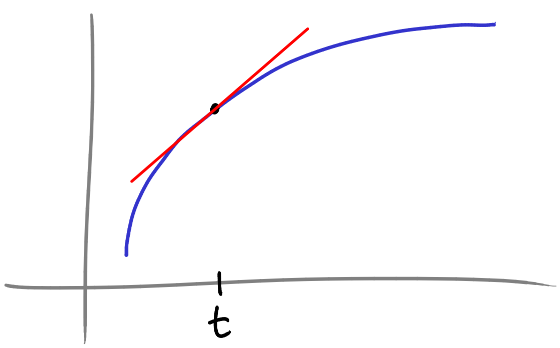
3 Sep 2015

Last time:



When the graph is a straight line, the velocity of the car is the slope of the line.

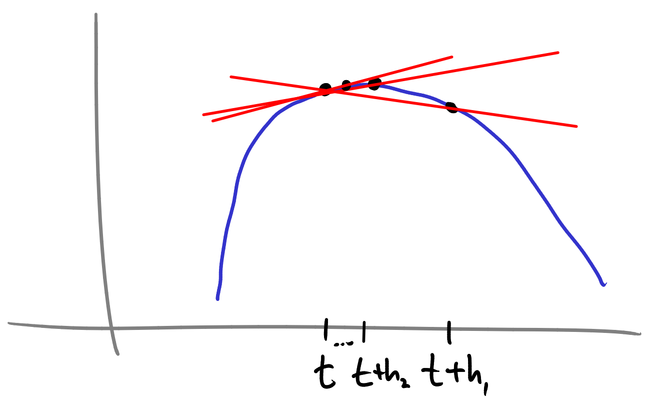
How to calculate (or even define) the velocity of the car, when the graph is not a straight line?



Try to say it's the slope of a tangent line to the graph.

Key idea: identify the tangent line as limit of secant lines

As we take h smaller and smaller, the "secant lines" through $(t, f(t))$ and $(t+h, f(t+h))$ are approaching a single fixed line — that's what we call the tangent line to the graph at $(t, f(t))$.



We need to learn how to do calculations which formalize this idea!

Limits

Suppose we have a function f,

and as x gets close to a (but not equal to a),

$f(x)$ gets close to L.

Then, we say $\lim\limits_{x \to a} f(x) = L$.

Ex

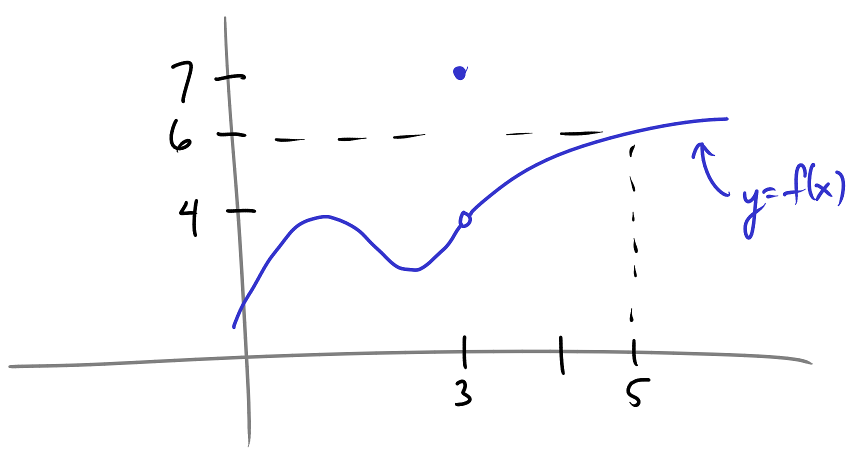
x	$f(x)$
1.0000	-6.000000
1.5000	-7.1250000
1.8000	-6.7680000
1.9000	-6.4410000
1.9900	-6.04940100
1.9990	-6.00499400
1.9999	-6.00049994
2.0001	-5.99949994
2.0010	-5.99499400
2.0100	-5.94939900
2.1000	-5.4390000
2.2000	-4.7520000
2.5000	-1.875000
3.0000	6.000000

2 is not in the domain of f

This table looks as if $\lim\limits_{x \to 2} f(x) = -6$.

(NB: $f(x)$ approaches -6 when x approaches 2 from either direction — either above or below.)

Ex



$\lim\limits_{x \to 3} f(x) = 4$

$f(3) = 7$

$\lim\limits_{x \to 5} f(x) = 6$

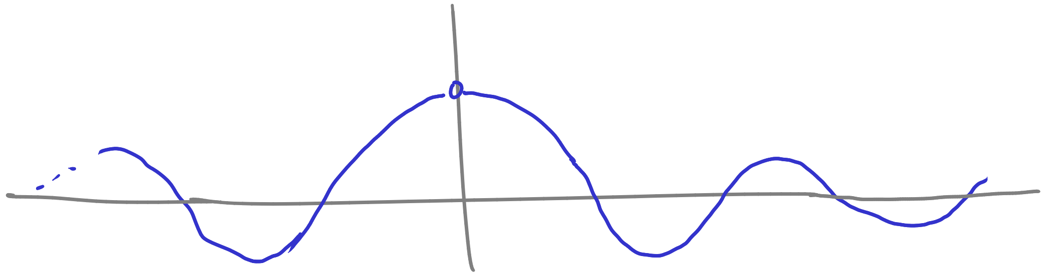
$f(5) = 6$

Ex $f(x) = \dfrac{\sin x}{x}$

x	f(x)
-1.0000	0.8414710
-0.50000	0.9588511
-0.20000	0.9933467
-0.10000	0.9983342
-0.010000	0.9999833
-0.0010000	0.9999998
0.0010000	0.9999998
0.010000	0.9999833
0.10000	0.9983342
0.20000	0.9933467
0.50000	0.9588511
1.0000	0.8414710

looks like $\lim_{x \to 0} f(x) = 1$ i.e $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$

(It's indeed true, we'll prove it later.)



Ex What is $\lim_{x \to 1} \dfrac{x-1}{x^3-1}$?

$$= \lim_{x \to 1} \frac{\cancel{x-1}}{\cancel{(x-1)}(x^2+x+1)}$$

$$= \lim_{x \to 1} \frac{1}{1+x+x^2}$$

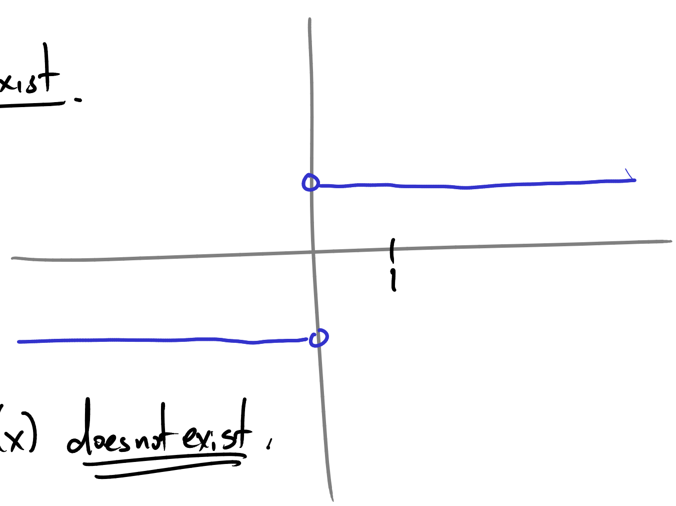
$$= \frac{1}{1+1+1^2} = \frac{1}{3}$$

x	
.9	1/3.7
.99	.336
.999	.3336
1.001	.3330 --

$\longrightarrow$ 1/3 ?

Sometimes $\lim_{x \to a} f(x)$ doesn't exist.

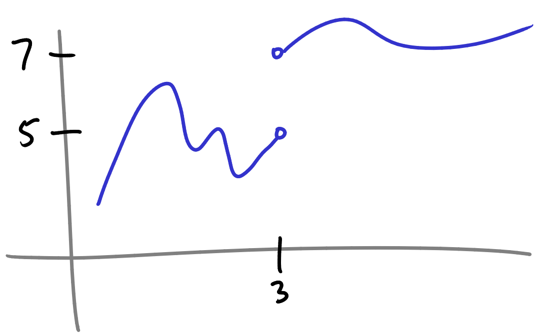
e.g. $f(x) = \dfrac{|x|}{x}$



$\lim_{x \to 1} f(x) = 1$ but $\lim_{x \to 0} f(x)$ doesn't exist.

But--

One-sided limits



Suppose we have a function f and as x gets close to a from the negative direction, $f(x)$ gets close to L.

Then, we say $\lim_{x \to a^-} f(x) = L$.

e.g. here, $\lim_{x \to 3^-} f(x) = 5$

Similarly define $\lim_{x \to a^+} f(x)$

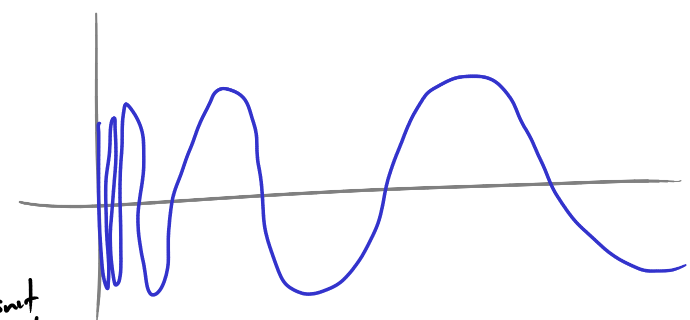
e.g. here, $\lim_{x \to 3^+} f(x) = 7$

Fact $\lim_{x \to a} f(x) = L$ just if $\left(\lim_{x \to a^+} f(x) = L \text{ and } \lim_{x \to a^-} f(x) = L \right)$.

Ex $f(x) = \sin\left(\frac{1}{x}\right)$

$\lim_{x \to 0} f(x)$ doesn't exist

– even $\lim_{x \to 0^+} f(x)$ doesn't exist!



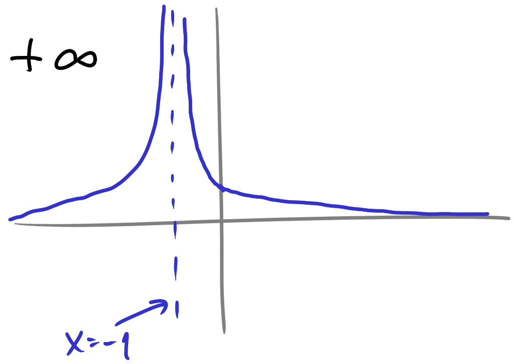
(and similarly $\lim_{x \to 0^-} f(x)$)

Limits of $\pm\infty$

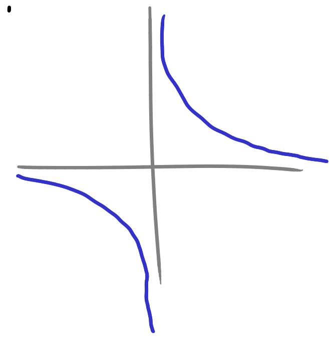
Suppose we have a function f and as $x \to a$, $f(x)$ grows without bound in the positive direction. Then, we say $\lim_{x \to a} f(x) = +\infty$,

Similarly, if as $x \to a$, $f(x)$ grows without bound in the negative direction, then we say $\lim_{x \to a} f(x) = -\infty$.

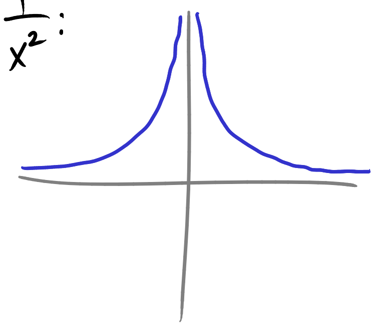
Ex $\lim_{x\to -1} \frac{1}{(x+1)^2} = +\infty$



$f(x) = \frac{1}{x}$:



$f(x) = \frac{1}{x^2}$:



(To see this w/o drawing the graph:
Imagine plugging in x very close to -1, say $x = -1.000001$

then $\frac{1}{(x+1)^2} = \frac{1}{(\text{very small})^2} = \frac{1}{(\text{very very small } \underline{\text{positive}})}$

$= \text{very very large } \underline{\text{positive}}$)

Ex $\lim_{x\to -3} \frac{x}{(x+3)^2}$ plug in x close to -3: $\frac{(\text{close to } -3)}{(\text{very small})^2}$

$= (\text{very big } \underline{\text{negative}})$

so $\lim_{x\to -3} \frac{x}{(x+3)^2} = -\infty$

Ex $\lim_{x\to 2} \frac{x-2}{x^2-4}$ plug in x close to 2: $\frac{(\text{very small})}{(\text{very small})} = ???$

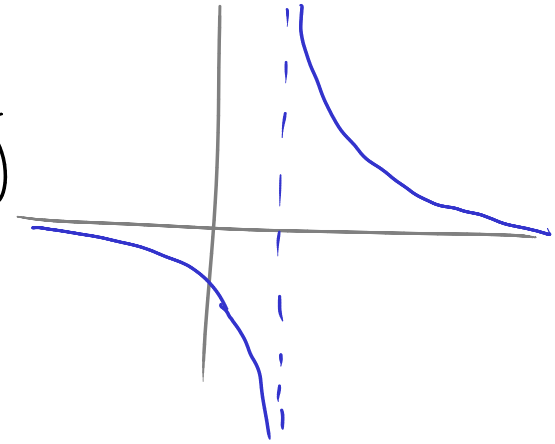
so need to do something clever:

$= \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)}$

$= \lim_{x\to 2} \frac{1}{x+2} = \underline{\underline{\frac{1}{4}}}$

$\underline{\text{Ex}}$ $\lim_{x\to 1} \frac{1}{x-1}$:

plug in x very close to 1: $\frac{1}{(\text{very small, } +\text{ve if } x>1,\ -\text{ve if } x<1)}$



So, $\lim_{x\to 1^+} \frac{1}{x-1} = +\infty$

$\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$

So $\lim_{x\to 1} \frac{1}{x-1}$ does not exist.

Limit Laws

Suppose c is any constant, and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

Then:

1. $\lim_{x\to a} \left(f(x)+g(x)\right) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} \left(f(x)-g(x)\right) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} c\cdot f(x) = c\cdot \lim_{x\to a} f(x)$
4. $\lim_{x\to a} \left(f(x)\cdot g(x)\right) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ (<u>if</u> $\lim_{x\to a} g(x) \neq 0$!)

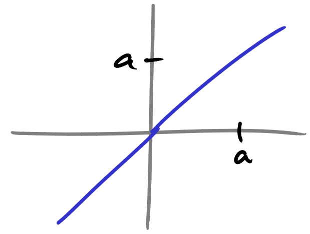
$\underline{\text{Ex}}$ If $\lim_{x\to 3} f(x) = 7$ and $\lim_{x\to 3} g(x) = 8$ then $\lim_{x\to 3} f(x)g(x) = 7\times 8 = 56$

6. $\lim_{x\to a} f(x)^n = \left(\lim_{x\to a} f(x)\right)^n$

$\underline{\text{Ex}}$ $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{131} = 1^{131} = 1$

(7) $\lim_{x \to a} c = c$

(8) $\lim_{x \to a} x = a$



Ex $\lim_{x \to 4} x^2 + 3x = \left(\lim_{x \to 4} x\right)^2 + 3 \cdot \left(\lim_{x \to 4} x\right)$

$= 4^2 + 3 \cdot 4 = \underline{\underline{28}}$

Ex $\lim_{x \to 2} \frac{x+2}{x^2-6} = \frac{\lim_{x \to 2}(x+2)}{\lim_{x \to 2}(x^2-6)} = \frac{2+2}{2^2-6} = \underline{\underline{-2}}$

in general, <u>Direct Substitution Property</u>. if $f(x)$ is a polynomial or a rational function, $\lim_{x \to a} f(x) = f(a)$. (*except* if this gives $\frac{0}{0}$)