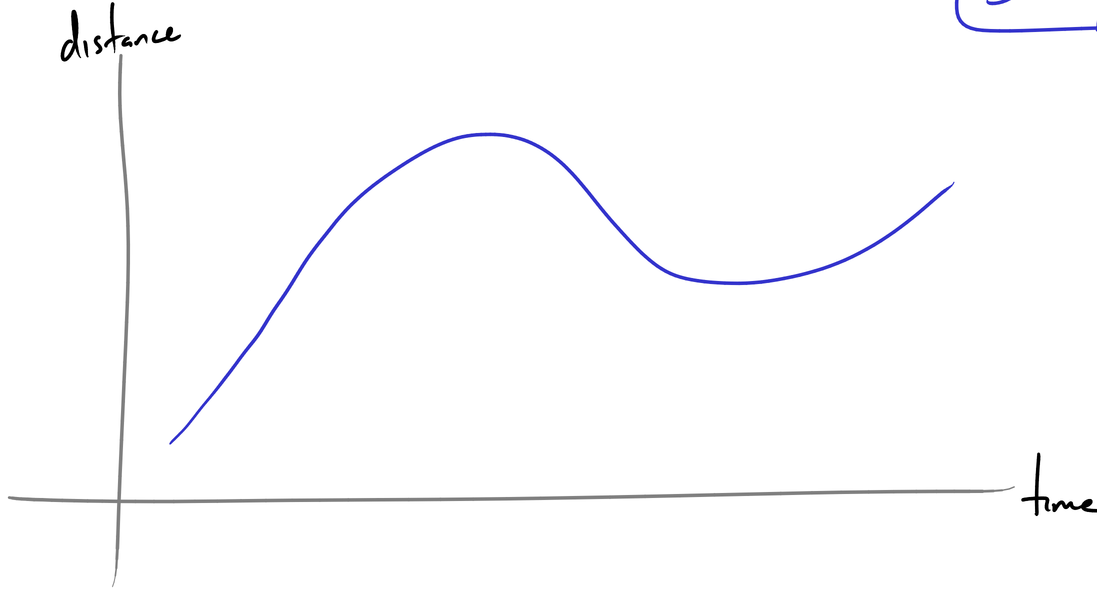
Lecture 3

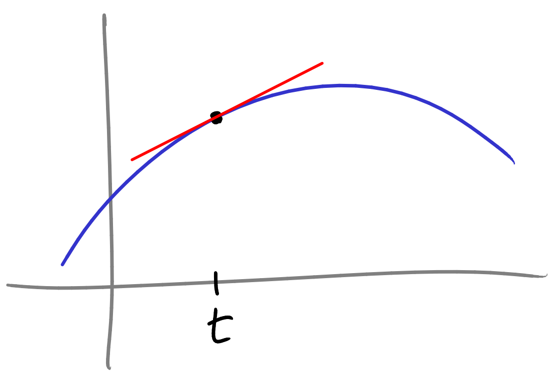
3 Sep 2015

Last time:

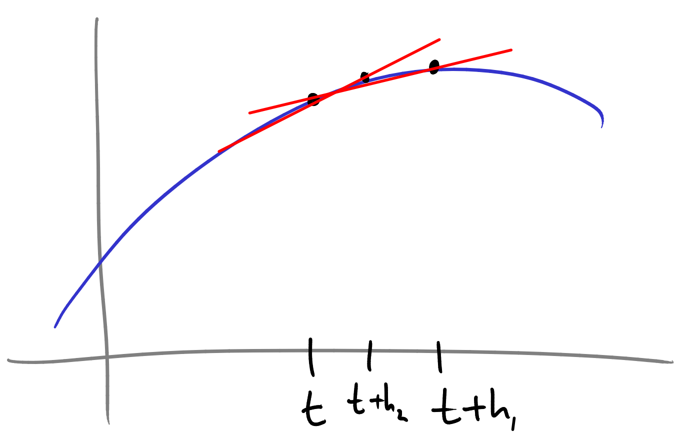


When the graph is a straight line, the velocity of the car is equal to the slope of that line.

When it's not a straight line, want to say the velocity at time t is equal to the slope of a tangent line to the graph at $(t, f(t))$.



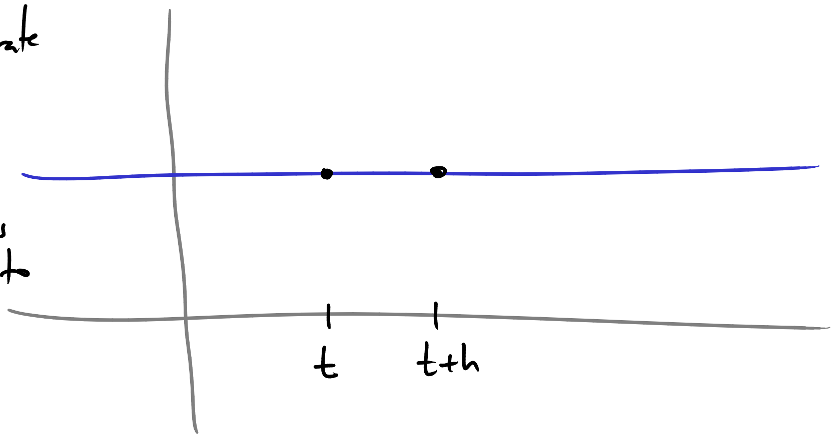
How to define "tangent line"? Key idea: look at it as a limit of secant lines



As h gets very small (but not equal to 0), the secant line through $(t, f(t))$ and $(t+h, f(t+h))$

approaches the sought-for tangent to the graph at $(t, f(t))$.

a degenerate example: here the secant lines are equal to the graph!



Limits

Suppose we have a function $f(x)$ and, as x gets close to a, $f(x)$ gets close to L. (a might not be in the domain of f)

Then, we say $\lim\limits_{x \to a} f(x) = L$.

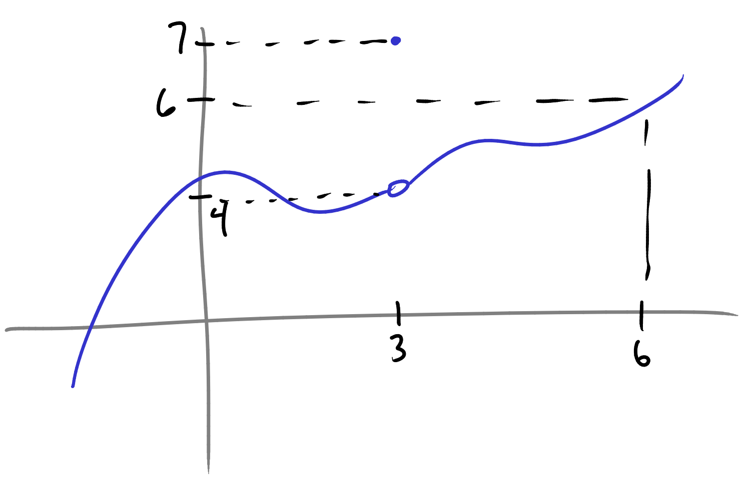
Ex

x	$f(x)$
1.0000	-6.000000
1.5000	-7.1250000
1.8000	-6.7680000
1.9000	-6.4410000
1.9900	-6.04940100
1.9990	-6.00499400
1.9999	-6.00049994
2.0001	-5.99949994
2.0010	-5.99499400
2.0100	-5.94939900
2.1000	-5.4390000
2.2000	-4.7520000
2.5000	-1.875000
3.0000	6.000000

This looks like $\lim\limits_{x \to 2} f(x) = -6$.

NB: this has nothing to do with what $f(2)$ is! (2 might not even be in the domain of f.)

Ex



$\lim\limits_{x \to 3} f(x) = 4$

$f(3) = 7$

$\lim\limits_{x \to 6} f(x) = 6$

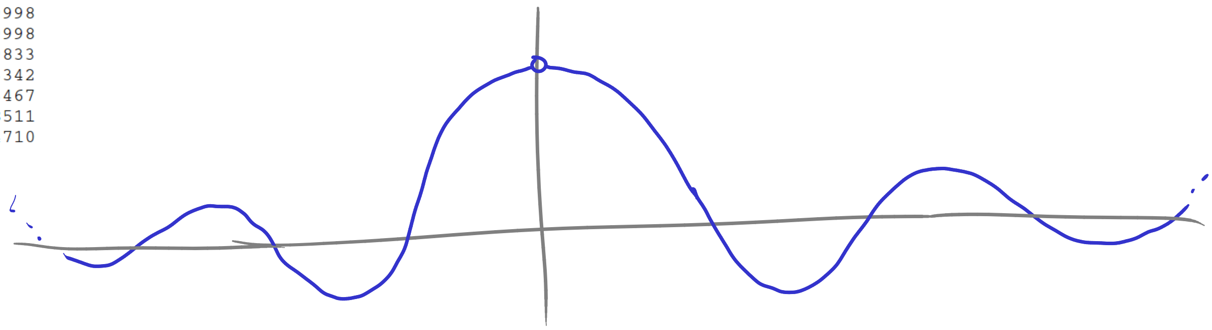
$f(6) = 6$

Ex $\lim_{x \to 0} \frac{\sin x}{x}$

x	$f(x)$
-1.0000	0.8414710
-0.50000	0.9588511
-0.20000	0.9933467
-0.10000	0.9983342
-0.010000	0.9999833
-0.0010000	0.9999998
0.0010000	0.9999998
0.010000	0.9999833
0.10000	0.9983342
0.20000	0.9933467
0.50000	0.9588511
1.0000	0.8414710

Looks like $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

(We'll prove later that it's really true.)



Ex What is $\lim_{x \to 1} \frac{x-1}{x^3-1}$?

x	$f(x)$
.9999	.3333666---
1.0001	.3333000---

$\rightsquigarrow \frac{1}{3}$

$$= \lim_{x \to 1} \frac{x-1}{(x-1)(x^2+x+1)}$$

$$= \lim_{x \to 1} \frac{1}{x^2+x+1}$$

for x close to 1, $\frac{1}{x^2+x+1} = \frac{1}{(\text{very close to } 3)}$

So indeed, $\lim_{x \to 1} \frac{x-1}{x^3-1} = \frac{1}{3}$

How to do w/o a calculator? "plugging in"

Say $x = 1 + \varepsilon$ ε very small

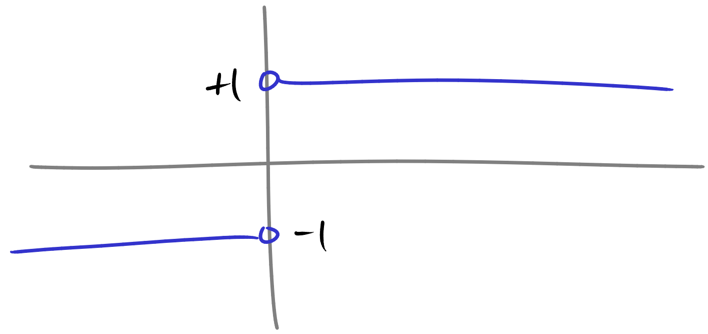
then, $\frac{x-1}{x^3-1} = \frac{(1+\varepsilon)-1}{(1+\varepsilon)^3-1} = \frac{\varepsilon}{1+3\varepsilon+3\varepsilon^2+\varepsilon^3-1}$

$$= \frac{\varepsilon}{3\varepsilon+3\varepsilon^2+\varepsilon^3}$$

$$\approx \frac{\varepsilon}{3\varepsilon} = \frac{1}{3}$$

Sometimes, $\lim_{x \to a} f(x)$ does not exist.

Ex $f(x) = \dfrac{|x|}{x}$



$\lim_{x \to 0} f(x)$ does not exist, because the value depends on which side we approach $x=0$ from.

If, as x gets close to a from the positive direction, $f(x)$ gets close to L, then we say $\lim_{x \to a^+} f(x) = L$.

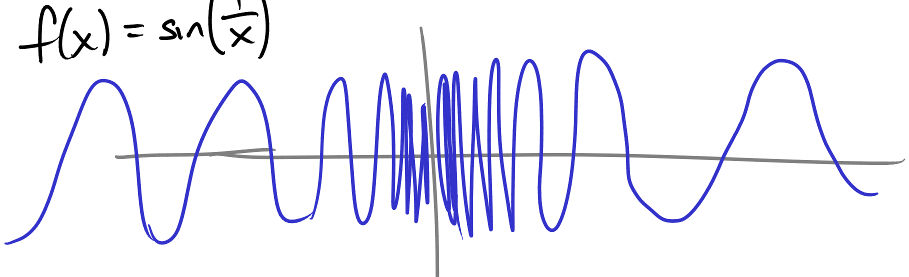
Similarly, if as x gets close to a from the negative direction, $f(x)$ gets close to L, then we say $\lim_{x \to a^-} f(x) = L$.

For $f(x) = \dfrac{|x|}{x}$, $\lim_{x \to 0^-} f(x) = -1$ $\lim_{x \to 0^+} f(x) = +1$

but $\lim_{x \to 0} f(x)$ doesn't exist.

Fact: $\lim_{x \to a} f(x) = L$ just if $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$.

Ex $f(x) = \sin\left(\frac{1}{x}\right)$



Here, $\lim_{x \to 0^+} f(x)$ doesn't exist, and $\lim_{x \to 0^-} f(x)$ doesn't exist.

(So, $\lim_{x \to 0} f(x)$ also doesn't exist.)

<u>Limits equaling $\pm\infty$</u>

Suppose we have a function f and as $x \to a$, $f(x)$ grows without bound in the positive direction. Then, we say $\lim_{x \to a} f(x) = +\infty$.

Similarly: if as $x \to a$, $f(x)$ grows without bound in the negative direction. Then we say $\lim_{x \to a} f(x) = -\infty$.

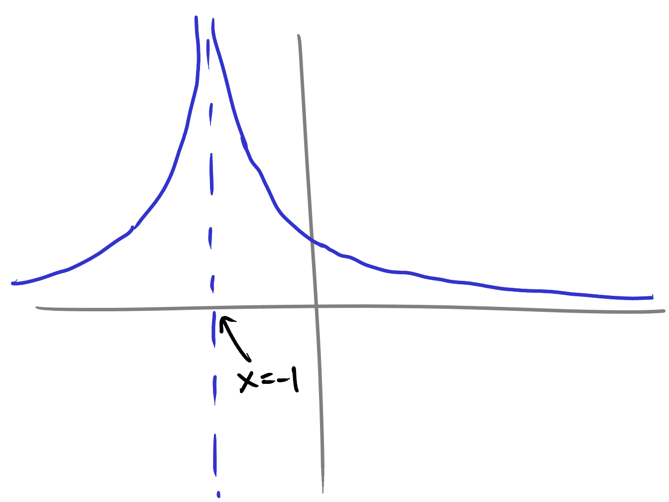
<u>Ex</u> $\lim_{x \to -1} \frac{1}{(x+1)^2} = ?$

$(x = -1 + \varepsilon)$

Imagine plugging in x very close to -1.

$$\frac{1}{(x+1)^2} = \frac{1}{(\text{very small})^2} = \frac{1}{(\text{very very small, positive})} = \text{very very big, positive}$$

So, $\lim_{x \to -1} \frac{1}{(x+1)^2} = +\infty$.



<u>Ex</u> $\lim_{x \to -3} \frac{x}{(x+3)^2} = ?$

Try plugging in x near -3: $\frac{x}{(x+3)^2} = \frac{(\text{near } -3)}{(\text{very small})^2} = (\text{very very big and negative})$

<u>Ex</u> $\lim_{x \to 2} \frac{x-2}{x^2-4} = \lim_{x \to 2} \frac{(x-2)}{(x-2)(x+2)} = \lim_{x \to 2} \frac{1}{x+2} = \frac{1}{4}$

(How did we know we needed to do the clever trick of factoring?

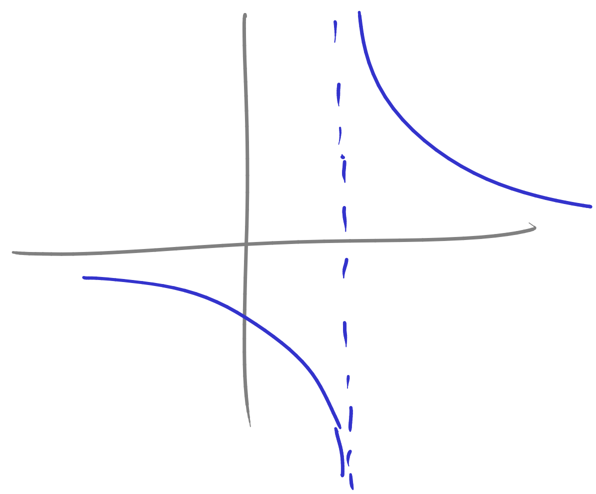
Try plugging in x near 2: $\dfrac{x-2}{x^2-4} = \dfrac{(\text{small})}{(\text{small})}$ and we have no way of knowing what that ratio is.)

<u>Ex</u> $\displaystyle\lim_{x\to 1} \frac{1}{x-1}$: plugging in x near 1, get $\dfrac{1}{x-1} = \dfrac{1}{(\text{very small})}$

or more precisely, $\dfrac{1}{\left(\text{very small and } \begin{cases}\text{positive if } x>1\\ \text{negative if } x<1\end{cases}\right)} = \left(\text{very big and } \begin{cases}\text{positive } x>1\\ \text{negative } x<1\end{cases}\right)$

So, $\displaystyle\lim_{x\to 1^+} \frac{1}{x-1} = +\infty$

$\displaystyle\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$



$\displaystyle\lim_{x\to 1} \frac{1}{x-1}$ <u>does not exist</u>.

<u>Limit Laws</u>

Suppose c is a constant, $\displaystyle\lim_{x\to a} f(x)$ exists, $\displaystyle\lim_{x\to a} g(x)$ exists. Then,

① $\displaystyle\lim_{x\to a} \big(f(x)+g(x)\big) = \Big(\lim_{x\to a} f(x)\Big) + \Big(\lim_{x\to a} g(x)\Big)$

② $\displaystyle\lim_{x\to a} \big(f(x)-g(x)\big) = \Big(\lim_{x\to a} f(x)\Big) - \Big(\lim_{x\to a} g(x)\Big)$

③ $\displaystyle\lim_{x\to a} c\cdot f(x) = c\cdot \lim_{x\to a} f(x)$

④ $\displaystyle\lim_{x\to a} f(x)\cdot g(x) = \Big(\lim_{x\to a} f(x)\Big)\cdot\Big(\lim_{x\to a} g(x)\Big)$

⑤ $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ <u>as long as</u> $\displaystyle\lim_{x\to a} g(x) \neq 0$.

Ex If $\lim_{x\to 3} f(x) = 7$ $\lim_{x\to 3} g(x) = 8$ then $\lim_{x\to 3} f(x)\cdot g(x) = 7\cdot 8 = 56$

⑥ $\lim_{x\to a} f(x)^n = \left(\lim_{x\to a} f(x)\right)^n$

Ex $\lim_{x\to 0} \dfrac{x^{19}}{(\sin x)^{19}} = \left(\lim_{x\to 0} \dfrac{x}{\sin x}\right)^{19} = 1^{19} = 1.$