Lecture 6

15 Sep 2015

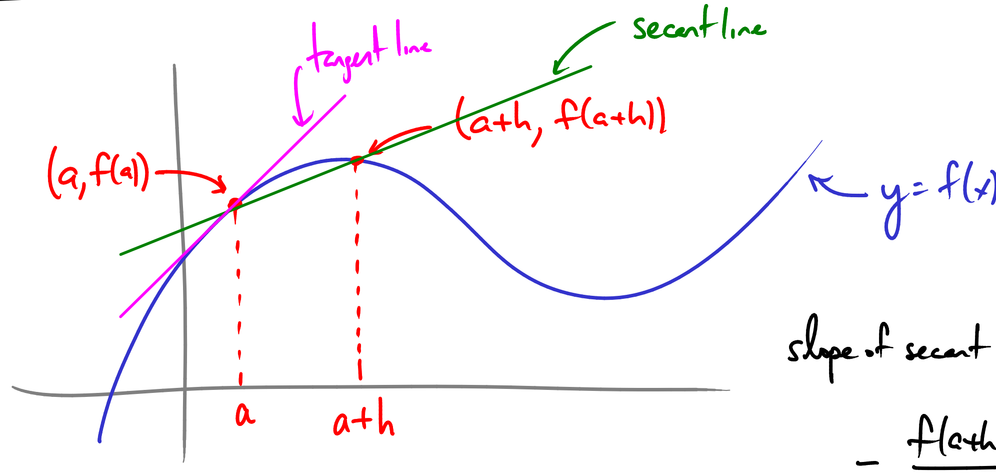
My office hr today 4–5:30 RLM 9.134

Last time: limits as $x \to \pm\infty$

Ex
$$\lim_{x\to\infty} \frac{3x^4 + \cancel{7x^2}^{\,0} + \cancel{7}^{\,0}}{9x^4 + \cancel{8x}^{\,0}} = \lim_{x\to\infty} \frac{3x^4}{9x^4} = \lim_{x\to\infty} \frac{3}{9} = \frac{1}{3}$$

$$\lim_{x\to-\infty} \frac{x^2 + \cancel{1}^{\,0}}{2x} = \lim_{x\to-\infty} \frac{x^2}{2x} = \lim_{x\to-\infty} \frac{x}{2} = -\infty$$

Derivatives



slope of secant line: $\frac{\text{rise}}{\text{run}}$

$$= \frac{f(a+h) - f(a)}{(a+h) - a}$$

$$= \frac{f(a+h) - f(a)}{h}$$

What we're after is slope of tangent line, not secant line. Idea: secant lines approach tangent line as we take $h \to 0$

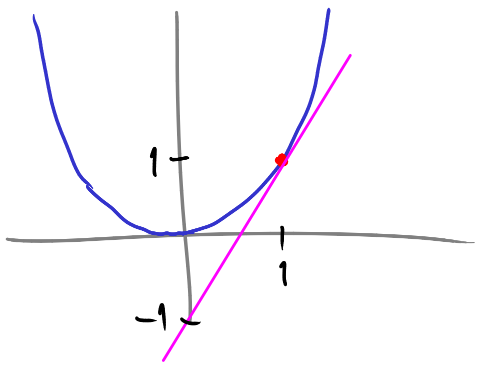
So, slope of tangent line at $(a, f(a))$ $= \lim_{h\to 0}$ (slope of secant line)

$$= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \quad \text{(if this limit exists)}$$

We call this slope the derivative: so we say the derivative of a function f at a point a is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \quad \text{if it exists.}$$

Ex What is the tangent line to the graph $y=x^2$ at $(1,1)$? $f(x)=x^2$



$$\text{slope} = f'(1)$$
$$= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0} \frac{(1+h)^2-1^2}{h}$$
$$= \lim_{h\to 0} \frac{1+2h+h^2-1}{h} = \lim_{h\to 0}\frac{2h+h^2}{h} = \lim_{h\to 0} 2+h = \underline{2}$$

So the tangent line passes thru $(1,1)$ and has slope 2:

so it's $y-1=2(x-1)$ ie $y-1=2x-2$ i.e. $\underline{y=2x-1}$

Ex If $f(x)=7x^2-3x+1$

① what is $f'(x)$?

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{(7(x+h)^2-3(x+h)+1)-(7x^2-3x+1)}{h}$$
$$= \cdots = \lim_{h\to 0}\frac{14xh-3h}{h} = 14x-3$$

② what is the tangent line to the graph $y=f(x)$ at $(x,y)=(-1,11)$?

slope is $f'(-1) = 14(-1)-3 = -17$

line with slope -17 thru $(-1,11)$ is

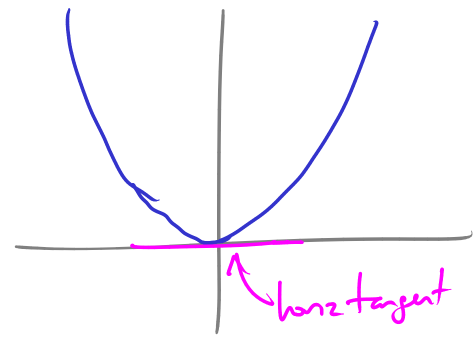
$$y-11 = -17(x-(-1))$$
$$y-11 = -17x-17$$
$$\underline{y=-17x-6}$$

Ex When does the graph of $y=x^2$ have a horizontal tangent line? $f(x)=x^2$

Horizontal tangent $\leftrightarrow$ slope $=0$ so need to solve $f'(x)=0$

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{2xh+h^2}{h}=\lim_{h\to 0}2x+h=\underline{\underline{2x}}$$

So $f'(x)=0$ when $2x=0$ ie $x=0$



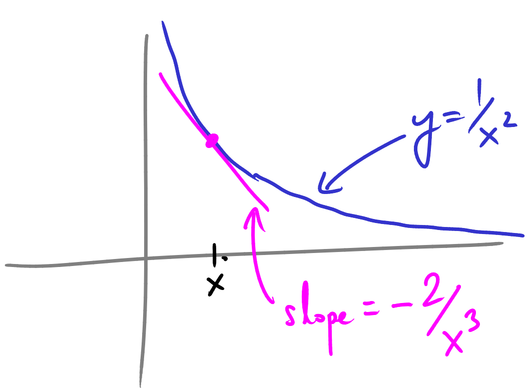
Ex If $f(x)=\frac{1}{x^2}$ what is $f'(x)$?

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}=\lim_{h\to 0}\frac{\left(\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right)}{h}$$

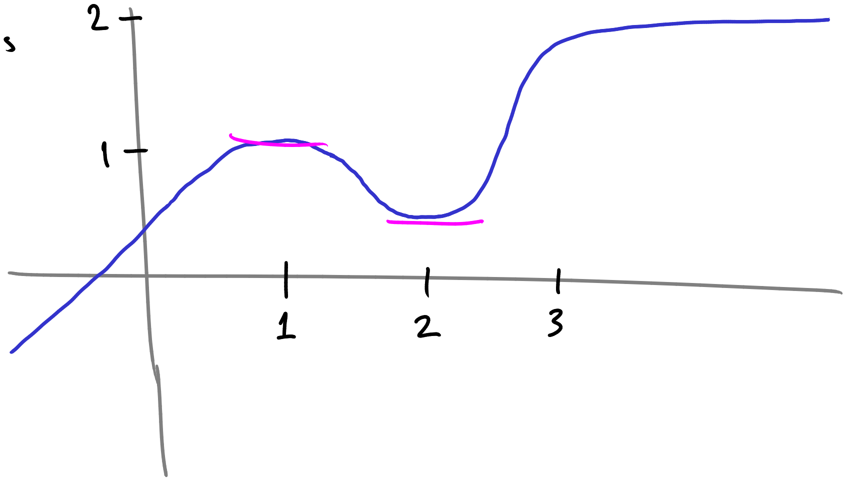
$$=\lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{h\cdot x^2\cdot(x+h)^2}=\lim_{h\to 0}\frac{-(2xh+h^2)}{h\cdot x^2\cdot(x+h)^2}=\lim_{h\to 0}\frac{-(2x+h)}{x^2(x+h)^2}$$

$$=\frac{-2x}{x^4}=\underline{\underline{\frac{-2}{x^3}}}$$

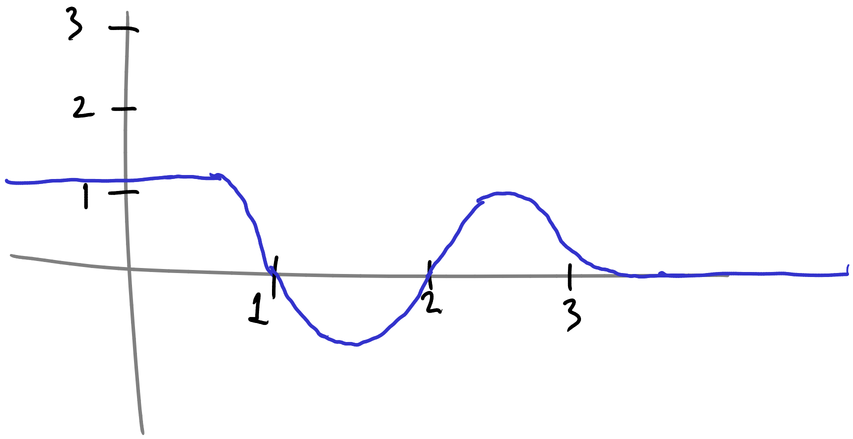
(Don't have to worry about $x=0$ since that's not in domain of f anyway)



Ex If $f(x)$ is

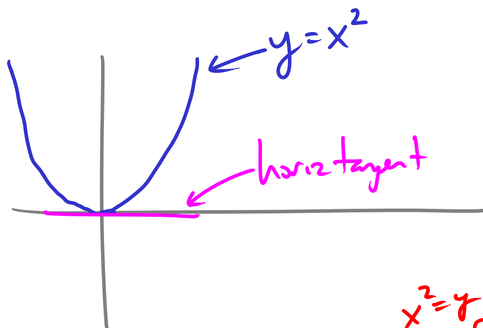


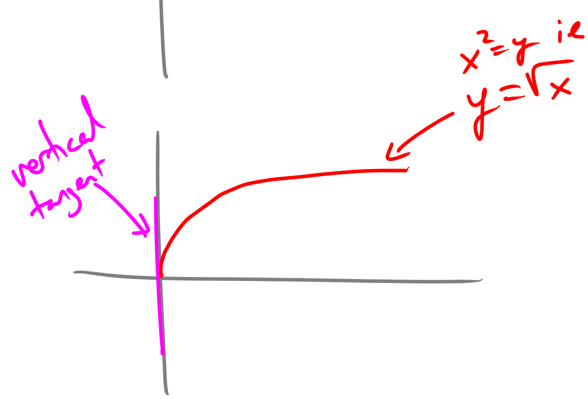
sketch $f'(x)$.



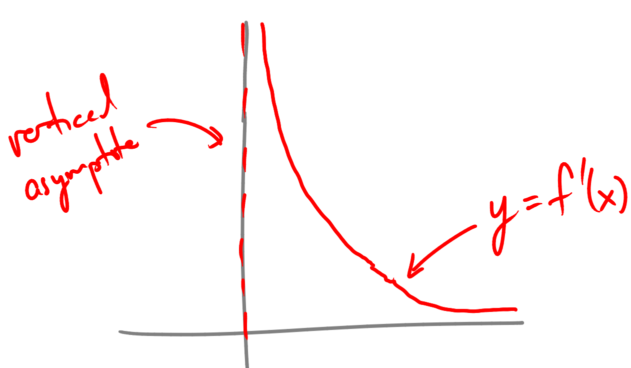
Ex If $f(x)=\sqrt{x}$

(1) sketch $f'(x)$.





$f(x)=\sqrt{x}$



② calculate $f'(x)$, $f(x)=\sqrt{x}$

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$=\lim_{h\to 0}\frac{(x+h)-x}{h\cdot(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

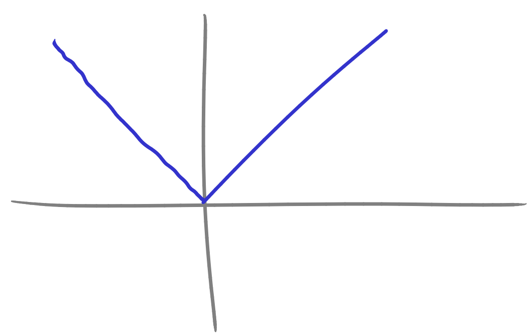
We say $f(x)$ is <u>differentiable at a</u> if $f'(a)$ exists

(ie $\lim_{h\to 0}\frac{f(a+h)-f(h)}{h}$ exists)

<u>Ex</u> $f(x)=x^2$ is differentiable at all real #'s a,

because $\lim_{h\to 0}\frac{f(a+h)-f(h)}{h}$ exists (and equals $2a$)

<u>Ex</u> Where is $f(x)=|x|$ differentiable?

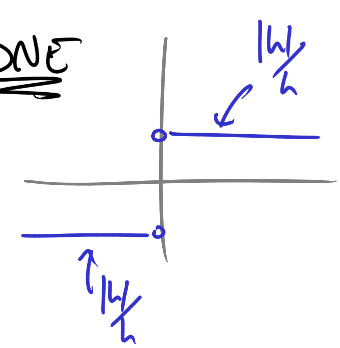


If $x>0$, $f'(x)=1$ (exists)

If $x<0$, $f'(x)=-1$

If $x=0$, let's look closer:

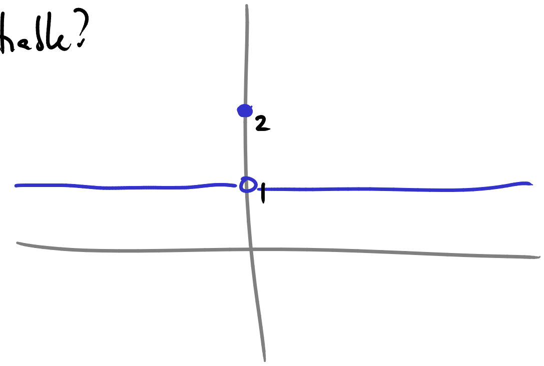
$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|-|0|}{h}=\lim_{h\to 0}\frac{|h|}{h}\ \text{DNE}$$



So f is <u>not</u> differentiable at $x=0$.

(In general, sharp corner $\Rightarrow$ not differentiable.)

Ex Where is $f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$ differentiable?



At any $x \neq 0$, $f'(x) = 0$.

At $x = 0$, $f'(x) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$

plug in small value of h: $\frac{f(h) - f(0)}{h} = \frac{1-2}{h} = \frac{-1}{h}$

→ the limit as $h \to 0$ DNE.

So $f(x)$ is not differentiable at $x = 0$.

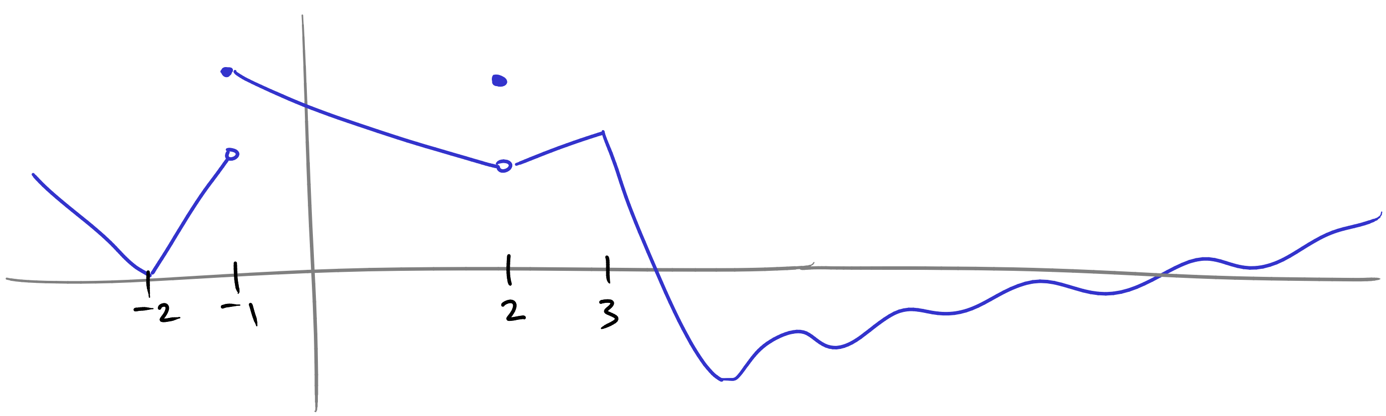
⟶ $f(x)$ is diff'ble at all x except 0.

In general: where f is not continuous, f is not differentiable.

Remark: at points where $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \pm\infty$

we say f is not differentiable.

Ex



f is differentiable except at $x = -2, -1, 2,$ or 3.

Interpretation of $f'(x)$

① If $x(t)$ is the position of an object at time t
then $x'(t)$ is the velocity of the object.
$\quad\; \parallel$
$\quad v(t)$

Ex An electron in a uniform electric field moves with $x(t) = \frac{1}{2}t^2$

What is its velocity at time t? $v(t) = x'(t) = \lim_{h \to 0} \frac{\frac{1}{2}(t+h)^2 - \frac{1}{2}t^2}{h} = \cdots = t$

② In general if t = time
$f'(t)$ is the rate of change of $f(t)$.

Ex If $V(t)$ = volume of water in Lake Travis (in gal)
$V'(t)$ = rate of change of the volume (in gal/sec)