Lecture 7

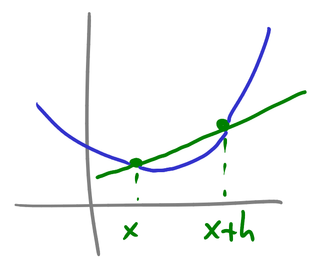
17 Sep 2015

- Exam 1 Sep 29 (week from Tue)
  - only need pencils + erasers
  - no calculators
  - problems very similar to QUEST HW ≈ 20 of them
  - covers all material from any lecture before the exam day
  - during class time

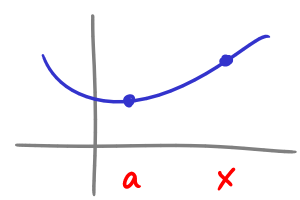
- I like "Problems Plus" # 3, 4, 5, 8 pp. 170–171

Last time: derivatives

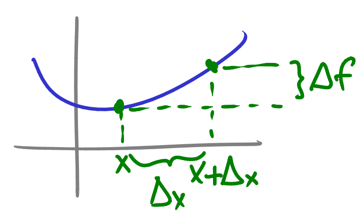
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



or: $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



or: $$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$



We can also repeat:

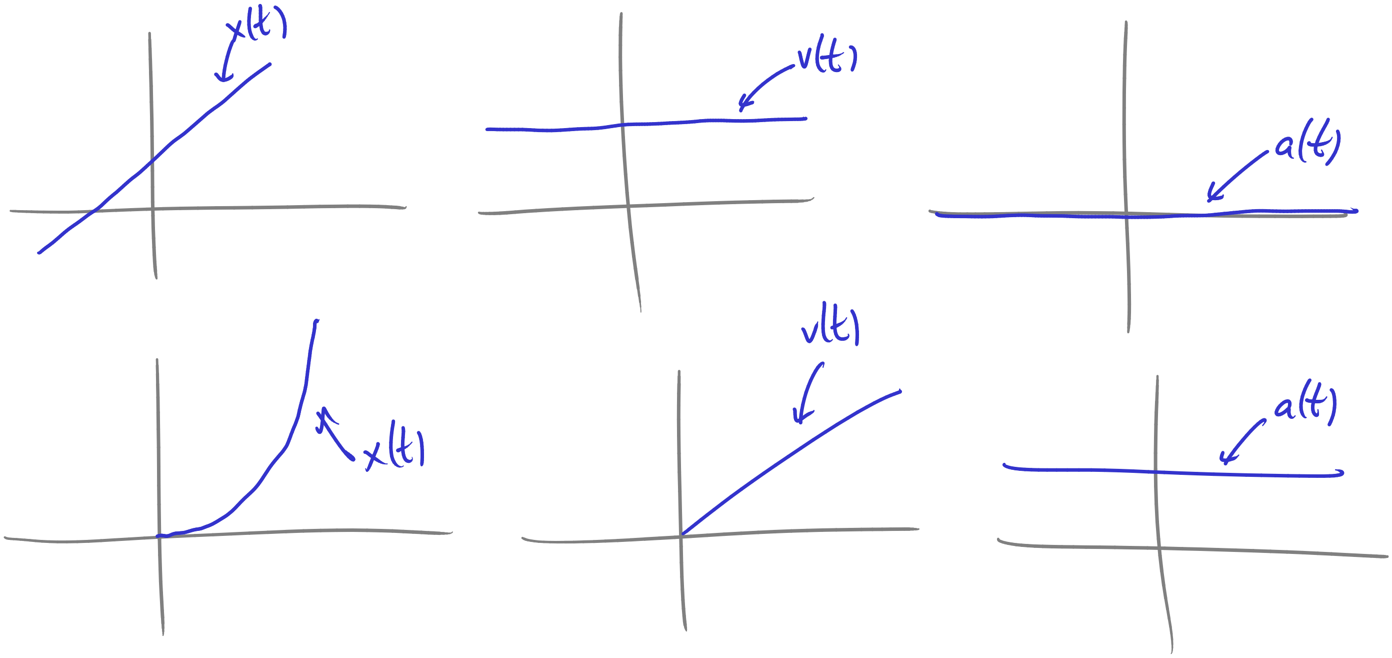
$$f''(x) = \text{derivative of } f'(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

"second derivative"

If $x(t)$ is position

$v(t) = x'(t)$ is velocity

$a(t) = v'(t) = x''(t)$ is acceleration — rate of change of the velocity



Another notation:

$$\frac{df}{dx} \quad \text{or} \quad \frac{d}{dx}f(x) \quad \text{mean} \quad f'(x)$$

$$\frac{d^2f}{dx^2} \quad \text{or} \quad \frac{d^2}{dx^2}f(x) \quad \text{mean} \quad f''(x)$$

$$\vdots$$

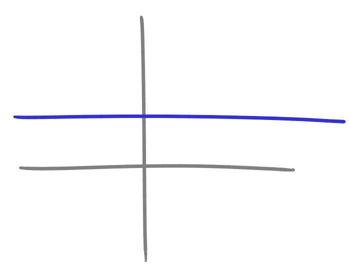
(n derivatives)

$$\frac{d^nf}{dx^n} \quad \text{or} \quad \frac{d^n}{dx^n}f(x) \quad \text{mean} \quad f'''''(x) = f^{(n)}(x)$$

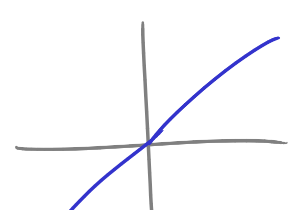
<u>Ex</u> $\frac{d}{dx}(x^2) = 2x \qquad \frac{d^2}{dx^2}(x^2) = \frac{d}{dx}(2x) = 2$

<u>Calculating derivatives</u>

Recall: $\frac{d}{dx}(c) = 0$



$\frac{d}{dx}(x) = 1$



Fact $\frac{d}{dx}(x^n) = nx^{n-1}$

Why? If $f(x) = x^n$

$$f'(a) = \lim_{x\to a} \frac{x^n - a^n}{x-a}$$

And $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})$

(e.g. $x^4 - a^4 = (x-a)(x^3 + x^2a + xa^2 + a^3)$)

So
$$\lim_{x\to a}\frac{x^n-a^n}{x-a} = \lim_{x\to a}\frac{(x-a)(x^{n-1}+x^{n-2}a+\cdots+a^{n-1})}{x-a}$$
$$= \lim_{x\to a} x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + a^{n-1}$$
$$= \underbrace{a^{n-1} + a^{n-1} + a^{n-1} + \cdots + a^{n-1}}_{n \text{ terms}}$$
$$= n\cdot a^{n-1}$$

Ex $\frac{d}{dx}(x^4) = 4x^3$ $\qquad$ $\frac{d}{dt}(t^9) = 9t^8$

Actually this works for any exponent:

Power rule $\frac{d}{dx}(x^r) = rx^{r-1}$ for r any constant

Ex $\frac{d}{dx}\sqrt{x} = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

Ex $\frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{d}{dx}(x^{-3}) = -3x^{-3-1} = -3x^{-4} = \frac{-3}{x^4}$

Constant Multiple Rule $\frac{d}{dx}(c\cdot f(x)) = c\cdot\frac{d}{dx}f(x)$

Why? $\lim_{h\to 0}\frac{c\cdot f(x+h) - c\cdot f(x)}{h} = c\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Ex $\frac{d}{dx}\left(\frac{-7}{x^2}\right) = -7\frac{d}{dx}\left(\frac{1}{x^2}\right) = -7\frac{d}{dx}(x^{-2}) = -7\cdot(-2x^{-3}) = \frac{14}{x^3}$

Ex $\frac{d}{dr}(\pi r^2) = \pi\frac{d}{dr}(r^2) = \pi\cdot 2r = 2\pi r$

area of a circle

circumference of a circle

(why?)

Sum Rule $\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$ (if both f, g are differentiable at x)

(exercise: prove this from definition of $\frac{d}{dx}$!)

Ex $\frac{d}{dx}\left(x+\frac{1}{x}\right) = \frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{1}{x}\right)$

$= 1 - \frac{1}{x^2}$ (Power Rule)

Ex $\frac{d}{dx}\left(\frac{3-5x^3}{\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{3}{\sqrt{x}} + \frac{-5x^3}{\sqrt{x}}\right)$

$= 3\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) - 5\frac{d}{dx}\left(\frac{x^3}{\sqrt{x}}\right)$ $x^3\cdot x^{-1/2}$

$= 3\frac{d}{dx}(x^{-1/2}) - 5\frac{d}{dx}(x^{5/2})$

$= 3\cdot\left(-\frac{1}{2}x^{-3/2}\right) - 5\cdot\left(\frac{5}{2}x^{3/2}\right)$

$= -\frac{3}{2}x^{-3/2} - \frac{25}{2}x^{3/2}$

Exponential functions

Fact There is a number, e, such that $\lim_{h\to 0}\frac{e^h-1}{h} = 1$.

($e \approx 2.71828\ldots$)

Fact $\frac{d}{dx}(e^x) = e^x.$ $\quad$ (not xe^{x-1})

$$\left[\text{Why? } \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h} = \lim_{h\to 0} e^x\left(\frac{e^h-1}{h}\right) = e^x \lim_{h\to 0}\frac{e^h-1}{h} = e^x\cdot 1 = e^x\right]$$

Ex $\frac{d}{dx}(x+7e^x) = 1+7e^x$

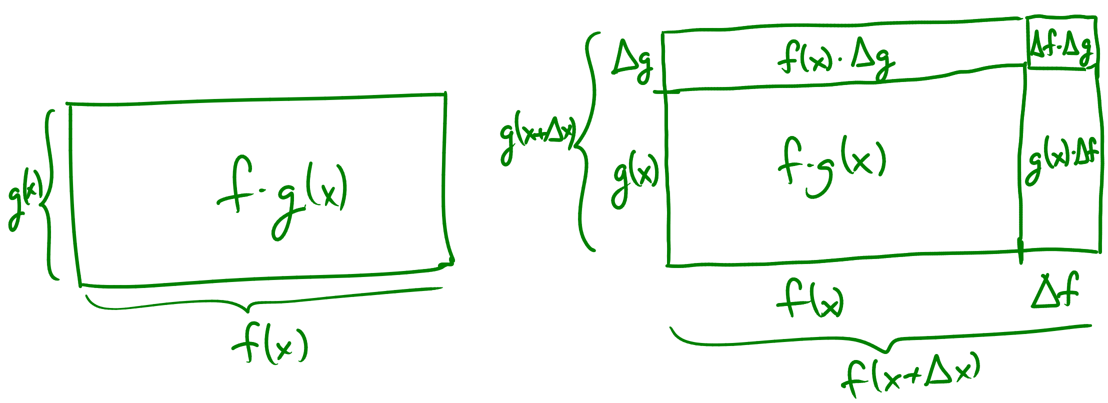
Product Rule $\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x)$ $\quad$ (NOT $f'(x)g'(x)$)

Ex $\frac{d}{dx}(\underbrace{x^3}_{f(x)}\underbrace{e^x}_{g(x)}) = \underbrace{3x^2}_{f'(x)}\underbrace{e^x}_{g(x)} + \underbrace{x^3}_{f(x)}\underbrace{e^x}_{g'(x)} = \underline{e^x(3x^2+x^3)}$

Ex $\frac{d}{dt}\big(\underbrace{\sqrt{t}}_{f}\underbrace{(a+bt)}_{g}\big) = \underbrace{\tfrac{1}{2}t^{-1/2}}_{f'}\underbrace{(a+bt)}_{g} + \underbrace{t^{1/2}}_{f}\underbrace{b}_{g'}$ $\quad$ a, b constants

$$= \underline{\tfrac{1}{2}at^{-1/2} + \tfrac{3}{2}bt^{1/2}}$$

[Why does it work? Want to calculate $\lim_{\Delta x\to 0}\frac{\Delta(fg)}{\Delta x}$

$f\times g =$  $= f\times g(x+\Delta x)$

ie $\Delta(fg) = f(x)\cdot\Delta g + g(x)\cdot\Delta f + \Delta f\cdot\Delta g$]

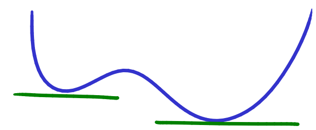
$$\frac{\Delta(fg)}{\Delta x} = f(x)\cdot\frac{\Delta g}{\Delta x} + g(x)\cdot\frac{\Delta f}{\Delta x} + \frac{\Delta f}{\Delta x}\cdot\Delta g$$

$$\frac{d}{dx}(fg) = \lim_{\Delta x\to 0}\left(\frac{\Delta(fg)}{\Delta x}\right) = f(x)\cdot\frac{dg}{dx} + g(x)\cdot\frac{df}{dx} + \cancel{\frac{df}{dx}\cdot\Delta g}^{\;0}$$

as we wanted!

Ex Where does the graph $y = \frac{e^x}{x}$ have a <u>horizontal tangent</u>?

Horiz tangents will occur when $\frac{dy}{dx} = 0$.



$$\frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{d}{dx}\left(e^x\cdot x^{-1}\right) = e^x\cdot x^{-1} + e^x\cdot\left(-\frac{1}{x^2}\right)$$

$$= e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)$$

<u>never</u> $=0$ $=0$ if $\frac{1}{x} = \frac{1}{x^2}$ i.e. $x = 1$