Lecture 8

22 Sep 2015

Midterm 1 next Tue (1 week from today)

Covers Lectures 1–8 — more precisely, the subset of lectures 1–8 which appeared in HW and can be done w/o calculator

Next HW will be due Fri not Wed, includes stuff from Lecture 8.

To prepare: ① go over HW's (each has 18 Quest-like problems)
② get extra probs from text sections we covered

My office hr: today 4–5:30 RLM 9.134

Last time: derivatives of polynomials, exponentials, product rule

Ex $\dfrac{d}{dx}\, \underset{f}{x^3}\,\underset{g}{e^x} = \underset{f'}{3x^2}\,\underset{g}{e^x} + \underset{f}{x^3}\,\underset{g'}{e^x}$

<u>Quotient Rule:</u> if f and g are differentiable at x and $g(x) \neq 0$
then $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$\frac{d}{dx}\left(\frac{\text{high}}{\text{low}}\right) = \frac{\text{low}\frac{d}{dx}\text{high} - \text{high}\frac{d}{dx}\text{low}}{\text{low}^2}$$

Ex $$\frac{d}{dx}\left(\frac{\overset{f}{1}}{\underset{g}{x}}\right) = \frac{\overset{g}{x}\cdot\overset{f'}{0} - \overset{f}{1}\cdot\overset{g'}{1}}{x^2} = -\frac{1}{x^2}$$

Ex $$\frac{d}{dx}\left(\frac{\overset{f}{x^2+3x+5}}{\underset{g}{1+x}}\right) = \frac{\overset{g}{(1+x)}\overset{f'}{(2x+3)} - \overset{f}{(x^2+3x+5)}\overset{g'}{(1)}}{\underset{g}{(1+x)^2}}$$

$$= \frac{(2x^2+5x+3) - (x^2+3x+5)}{(1+x)^2} = \underline{\underline{\frac{x^2+2x-2}{(1+x)^2}}}$$

Remark if instead we took $\frac{d}{dx}\left(\frac{x^2+3x+2}{1+x}\right) \leftarrow f(x)$

then by same kind of calculation, get $\frac{(2x^2+5x+3)-(x^2+3x+2)}{(1+x)^2} = \frac{x^2+2x+1}{(1+x)^2}$

$$= \frac{(1+x)^2}{(1+x)^2} = \underline{\underline{1}}$$

(why so simple? because actually $f(x) = \frac{(x+2)(x+1)}{x+1} = x+2$, and $\frac{d}{dx}(x+2) = 1$!) (for $x \neq -1$)

Ex $\frac{d}{dx}\left(\frac{e^x}{x-1}\right) = \frac{(x-1)e^x - e^x(1)}{(x-1)^2} = \underline{\underline{\frac{e^x(x-2)}{(x-1)^2}}}$

Ex $\frac{d}{dx}\left(\frac{e^x}{e^x+1}\right) = \frac{(e^x+1)e^x - e^x e^x}{(e^x+1)^2} = \frac{e^{2x}+e^x-e^{2x}}{(e^x+1)^2} = \underline{\underline{\frac{e^x}{(e^x+1)^2}}}$

Ex $\frac{d}{dx} \frac{e^x+e^x x^3}{\sqrt{x}}$ — could do this by quotient rule but easier to just split up:

$$= \frac{d}{dx}\left(\frac{e^x}{\sqrt{x}} + \frac{e^x x^3}{\sqrt{x}}\right)$$

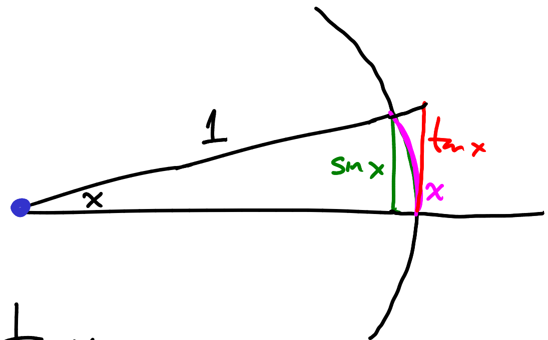
$$= \frac{d}{dx}\left(e^x x^{-1/2} + e^x x^{5/2}\right) = \cdots \quad \text{(use product rule)}$$

Derivatives of Trig Functions

Need to work out a few limits first.

Fact $\lim_{x\to 0} \frac{\sin x}{x} = 1.$

Why? $\sin x = \frac{\text{opposite}}{\text{hypotenuse}}$



circumf. of circle

length of arc

part of circle we cover

$$2\pi \cdot \frac{x}{2\pi} = x$$

From this picture: $\sin x < x < \tan x$

so $\sin x < x$ and $x < \tan x = \frac{\sin x}{\cos x}$

$\longrightarrow \frac{\sin x}{x} < 1$ $\qquad \cos x < \frac{\sin x}{x}$

ie $\cos x < \frac{\sin x}{x} < 1$

and $\lim_{x\to 0} \cos x = 1$, $\lim_{x\to 0} 1 = 1$, so Squeeze Theorem $\Rightarrow \lim_{x\to 0} \frac{\sin x}{x} = 1.$

Fact $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0.$

$$\text{Why? } \lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1}$$

$$= \lim_{x\to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)}$$

$\cos^2 x + \sin^2 x = 1$

$\cos^2 x - 1 = -\sin^2 x$

$$= \lim_{x\to 0} \frac{-\sin^2 x}{x(\cos x + 1)}$$

$$= \lim_{x\to 0} -\frac{\sin x}{x} \cdot \frac{\sin x}{\cos x + 1}$$

$$= (-1) \cdot \left(\frac{0}{2}\right) = 0$$

Fact $\frac{d}{dx}(\sin x) = \cos x.$

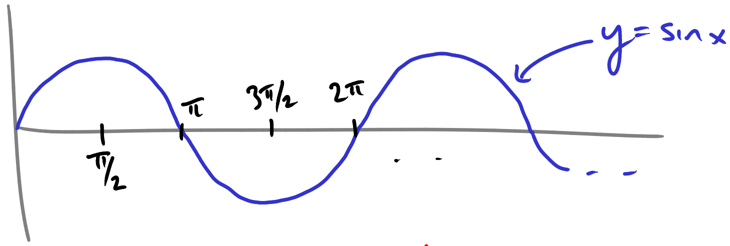
$$\text{Why? } \frac{d}{dx} \sin x = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

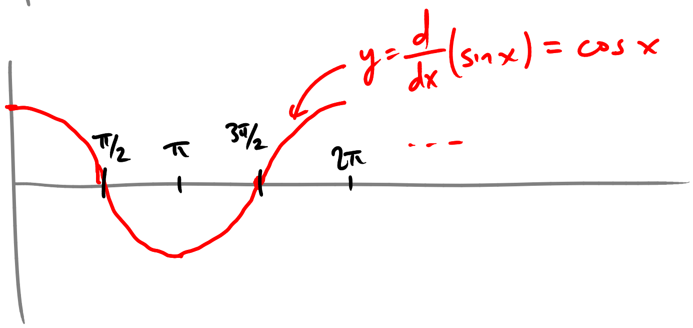
$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} (\sin x) \cdot \left(\frac{\cos h - 1}{h}\right) + (\cos x) \cdot \left(\frac{\sin h}{h}\right)$$

$$= \sin x \cdot \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \cos x \cdot \lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$

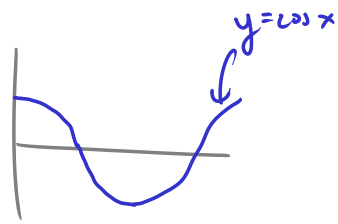
$$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x$$

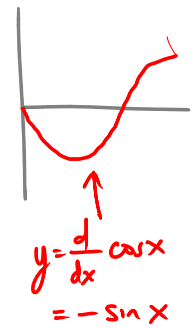




Ex $\frac{d}{dx}\left(x^4 \sin x\right) = 4x^3 \sin x + x^4 \cos x$.

Similarly, can derive





Fact $\frac{d}{dx}(\cos x) = -\sin x$.

All other derivatives of trig functions follow from these.

Ex $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$$= \frac{1}{\cos^2 x} = \underline{\underline{\sec^2 x}} \qquad \left(\sec x = \frac{1}{\cos x}\right)$$

(or: could simplify $\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \tan^2 x$ and then use identity $1 + \tan^2 x = \sec^2 x$)

Summary:

$\frac{d}{dx} \sin x = \cos x$ $\qquad \frac{d}{dx} \tan x = \sec^2 x$ $\qquad \frac{d}{dx} \sec x = \sec x \tan x$

$\frac{d}{dx} \cos x = -\sin x$ $\qquad \frac{d}{dx} \cot x = -\csc^2 x$ $\qquad \frac{d}{dx} \csc x = -\csc x \cot x$

Ex Where does the graph $y = \dfrac{\sec x}{1+\tan x}$ have horizontal tangent?

Need to solve $\dfrac{dy}{dx} = 0$.

$$\frac{dy}{dx} = \frac{(1+\tan x)\cdot \sec x \tan x - (\sec x)\cdot \sec^2 x}{(1+\tan x)^2}$$

$$= \frac{\sec x \tan x + \sec x \tan^2 x - \sec x \sec^2 x}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2} \qquad \begin{aligned} &\sec^2 x - \tan^2 x = 1 \\ &\text{so } \tan^2 x - \sec^2 x = -1 \end{aligned}$$

$$= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$$

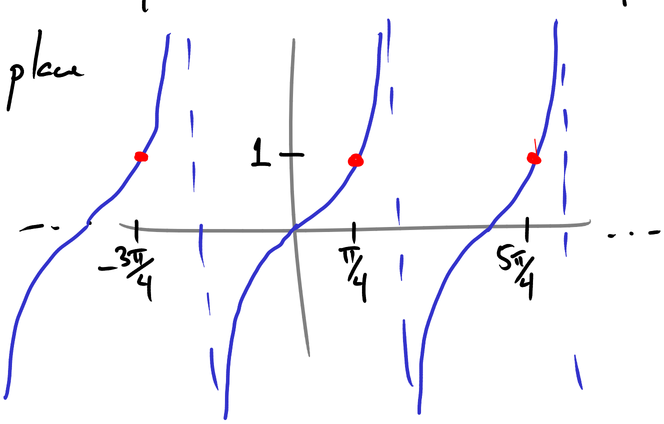
where is this $=0$? wherever $\tan x = 1$ $\left(\sec x = \frac{1}{\cos x} \text{ never } = 0\right)$

e.g. at $x = \pi/4$ since $\tan(\pi/4) = 1$

and $\tan x$ is periodic with period π, so $\tan(\pi/4 + n\pi) = \tan(\pi/4) = 1$ for any integer n

So, get horiz tangents at $\pi/4 + n\pi$ for all integers n (positive or negative)

(and not at any other place



)

Ex What is the 21^{st} derivative of $f(x) = \sin x + x^{15}$?

Taking derivatives:

$$f'(x) = \cos x + 15x^{14}$$
$$f''(x) = -\sin x + 15\cdot 14 x^{13}$$
$$f^{(3)}(x) = -\cos x + 15\cdot 14\cdot 13 x^{12}$$
$$f^{(4)}(x) = \sin x + 15\cdot 14\cdot 13\cdot 12 x^{11}$$

$$f^{(18)}(x) = \sin x + (\text{———})\, x^7$$

$$\vdots$$

$$f^{(20)}(x) = \sin x$$

$$f^{(21)}(x) = \underline{\underline{\cos x}}$$

(Remark: This "self-reproducing" behavior is a hint that trig and exponential functions are closely connected. Indeed, $e^{ix} = \cos x + i \sin x$)