Lecture 8

22 Sep 2015

Exam 1 next Tue (1 week)

- 18 questions, very like Quest HW
- no calculators
- covers material thru Lecture 8
- next HW due Fri next week, not Wed
- my office hr today 4–5:30 RLM 9.134

Last time: rules for computing derivatives (Power Rule, Product Rule, deriv. of exponentials)

Ex $\dfrac{d}{dx}\left(\underset{f}{\sqrt{x}}\,\underset{g}{e^x}\right) = \underset{f'}{\left(\dfrac{1}{2\sqrt{x}}\right)}\underset{g}{e^x} + \underset{f}{\sqrt{x}}\,\underset{g'}{e^x} = e^x\left(\dfrac{1}{2\sqrt{x}} + \sqrt{x}\right)$

How about $\dfrac{d}{dx}\left(\dfrac{4x^2+9x+7}{x+1}\right)$?

Fact (Quotient Rule). If f and g are differentiable at x, and $g(x) \neq 0$,

then $\dfrac{d}{dx}\dfrac{f(x)}{g(x)} = \dfrac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

$\left[\dfrac{d}{dx}\dfrac{\text{high}}{\text{low}} = \dfrac{\text{low}\frac{d}{dx}\text{high} - \text{high}\frac{d}{dx}\text{low}}{\text{low}^2}\right]$

Ex $\dfrac{d}{dx}\left(\dfrac{x^2+3x+3}{1+x}\right) = \dfrac{(1+x)\frac{d}{dx}(x^2+3x+3) - (x^2+3x+3)\frac{d}{dx}(1+x)}{(1+x)^2}$

$= \dfrac{(1+x)(2x+3) - (x^2+3x+3)(1)}{(1+x)^2}$

$= \dfrac{2x^2+5x+3 - (x^2+3x+3)}{(1+x)^2} = \dfrac{x^2+2x}{(1+x)^2}$

Remark: if we put a 2 instead of 3 in the numerator —

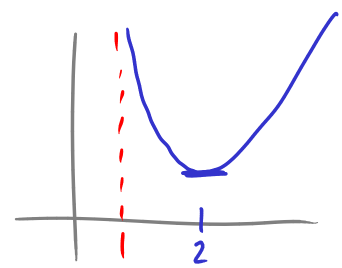
$$\frac{d}{dx}\left(\frac{x^2+3x+2}{1+x}\right) = \cdots = \frac{2x^2+5x+3-(x^2+3x+2)}{(1+x)^2} = \frac{x^2+2x+1}{(1+x)^2} = \frac{(1+x)^2}{(1+x)^2} = \underline{\underline{1}}$$

Why? Because actually $\frac{x^2+3x+2}{1+x} = \frac{(x+2)(x+1)}{1+x} = x+2$

and $\frac{d}{dx}(x+2) = 1$!

Ex $\frac{d}{dx}\left(\frac{\overset{f}{e^x}}{\underset{g}{x-1}}\right) = \frac{\overset{g}{(x-1)}\overset{f'}{e^x} - \overset{f}{e^x}\overset{g'}{(1)}}{(x-1)^2} = \frac{e^x(x-1-1)}{(x-1)^2} = \frac{e^x(x-2)}{(x-1)^2}$

(so graph of $y = \frac{e^x}{x-1}$ has horiz tangent at $x=2$)

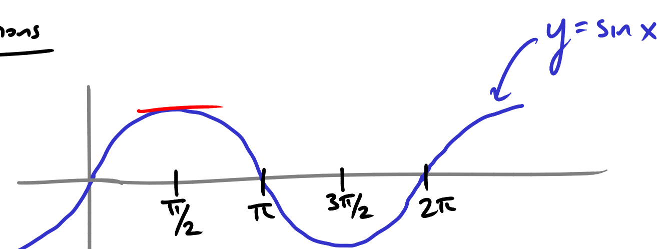


Ex $\frac{d}{dx}\left(\frac{1}{x}\right)$ in a complicated way: use quotient rule

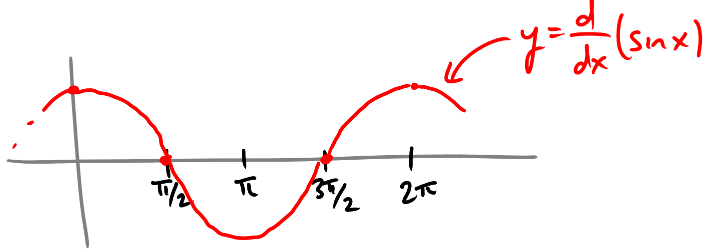
$$= \frac{x(0)-1\cdot 1}{x^2} = \underline{\underline{-\frac{1}{x^2}}}$$

Derivatives of Trig Functions

What is $\frac{d}{dx}(\sin x)$?



Just by looking at the graph, can get a rough picture of $\frac{d}{dx}\sin x$:



$\longrightarrow$ looks roughly like $\cos x$!

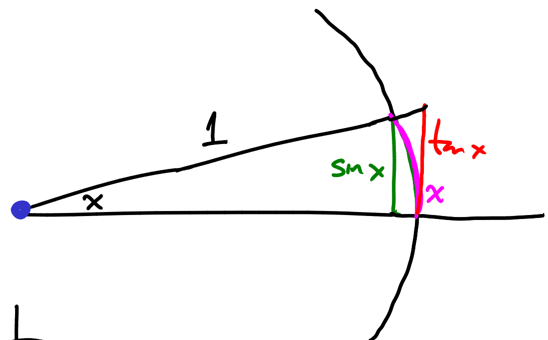
Indeed that's the right answer.

Fact $\dfrac{d}{dx}\sin x = \cos x.$

Why? First let's calculate some limits we will need:

Fact $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1.$

Why? $\sin x = \dfrac{\text{opposite}}{\text{hypotenuse}}$



circumference of whole circle

$\downarrow$

$2\pi\cdot\dfrac{x}{2\pi} = x$

$\uparrow$

part of circle we cover

From this picture: $\sin x < x < \tan x$

so $\sin x < x$ and $x < \tan x = \dfrac{\sin x}{\cos x}$

$\longrightarrow \dfrac{\sin x}{x} < 1$ $\qquad \cos x < \dfrac{\sin x}{x}$

ie $\cos x < \dfrac{\sin x}{x} < 1$

and $\displaystyle\lim_{x\to 0}\cos x = 1$, $\displaystyle\lim_{x\to 0} 1 = 1$, so Squeeze Theorem $\Rightarrow \displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1.$

Fact $\displaystyle\lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$

Why? $\dfrac{\cos x - 1}{x}\cdot\dfrac{\cos x + 1}{\cos x + 1} = \dfrac{\cos^2 x - 1}{x(\cos x + 1)} = \dfrac{-\sin^2 x}{x(\cos x + 1)}$

$\sin^2 x + \cos^2 x = 1$

$\cos^2 x - 1 = -\sin^2 x$

and $\displaystyle\lim_{x\to 0}\frac{-\sin^2 x}{x(\cos x + 1)} = \lim_{x\to 0}\left(-\frac{\sin x}{x}\right)\left(\frac{\sin x}{\cos x + 1}\right)$

$= (-1)\cdot\left(\dfrac{0}{2}\right) = 0.$

Now use these to compute $\dfrac{d}{dx}\sin x$:

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \cdot \left(\frac{\sin h}{h}\right)$$

$$= \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

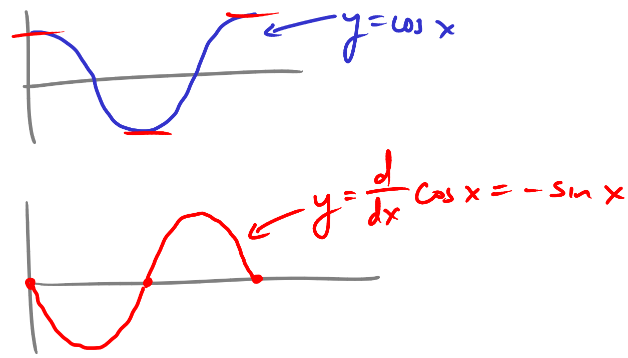
$$= \sin x \cdot 0 + \cos x \cdot 1$$

$$= \underline{\underline{\cos x}}$$

$\underline{\text{Ex}}$ $\frac{d}{dx}\left(x^4 \sin x\right) = 4x^3 \sin x + x^4 \cos x$

Similarly:

$\underline{\text{Fact}}$ $\frac{d}{dx}(\cos x) = -\sin x$



Other trig derivatives can be derived from these.

$\underline{\text{Ex}}$ $$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\overset{f}{\sin x}}{\underset{g}{\cos x}}\right) = \frac{\overset{g}{\cos x} \cdot \overset{f'}{\cos x} - \overset{f}{\sin x} \cdot \overset{g'}{(-\sin x)}}{\underset{g^2}{\cos^2 x}} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \underline{\underline{\sec^2 x}}$$

(We could also have simplified $\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}$
$= 1 + \tan^2 x$
$= \sec^2 x$)

Summary:

$\frac{d}{dx}\sin x = \cos x$ $\qquad$ $\frac{d}{dx}\tan x = \sec^2 x$ $\qquad$ $\frac{d}{dx}\sec x = \sec x \cdot \tan x$

$\frac{d}{dx}\cos x = -\sin x$ $\qquad$ $\frac{d}{dx}\cot x = -\csc^2 x$ $\qquad$ $\frac{d}{dx}\csc x = -\csc x \cot x$

(good idea to memorize these!)

Ex When does the graph $y = \frac{\sec x}{1+\tan x}$ have a horizontal tangent?

Horizontal tangent: $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = \frac{(1+\tan x)\frac{d}{dx}\sec x - (\sec x)\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{(1+\tan x)\cdot \sec x \tan x - \sec x \cdot \sec^2 x}{(1+\tan x)^2}$$

never 0

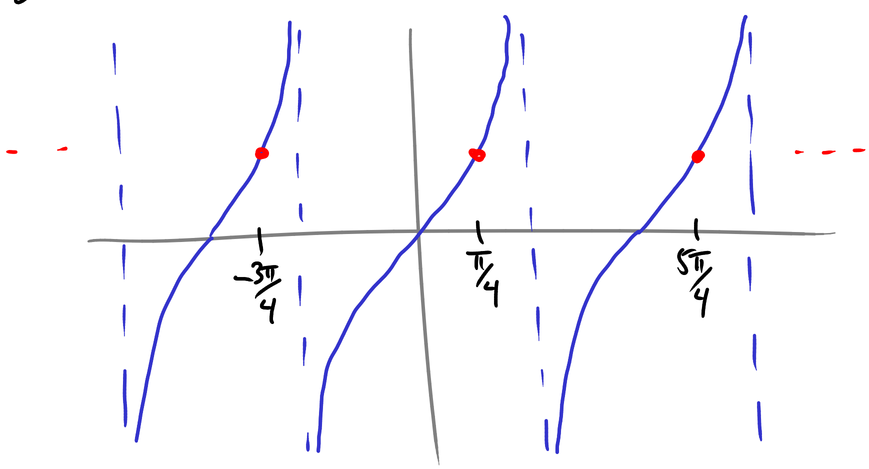
$$= \frac{\sec x \cdot (\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2}$$

$\sec^2 x - \tan^2 x = 1$

$$= \frac{\sec x \cdot (\tan x - 1)}{(1+\tan x)^2}$$

So, horizontal tangents only when $\tan x = 1$.

e.g. at $x = \pi/4$ since $\tan(\pi/4) = 1$.



In fact $\tan x = 1$ at $x = \frac{\pi}{4} + n\pi$ for n any integer — these are the horiz tangents for $y = \frac{\sec x}{1+\tan x}$

Ex If $f(x) = \sin x + x^{16}$

what is $f^{(21)}(x)$? (21st derivative)

21st deriv. $\dfrac{d^{21}}{dx^{21}}(x^{16}) = \underline{0}$

$\dfrac{d^{21}}{dx^{21}}(\sin x) = \cos x$

$$\begin{tikzcd}
 & \sin x \arrow[dr, "d/dx"] & \\
-\cos x \arrow[ur, "d/dx"] & & \cos x \arrow[dl, "d/dx"] \\
 & -\sin x \arrow[ul, "d/dx"] &
\end{tikzcd}$$