Lecture 9

24 Sep 2015

Midterm 1 Tue in class

Just need pencils, ID

Can't use calculators

Last time: trig limits and trig derivatives

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

What is $\lim_{x\to 0} \frac{\sin 3x}{3x}$?

Plug in small x, say $x = 0.001$, $\frac{\sin(0.003)}{0.003}$

ie get $\frac{\sin(\text{small \#})}{\text{same small \#}}$

just like in evaluating $\lim_{x\to 0} \frac{\sin x}{x}$!

So $\lim_{x\to 0} \frac{\sin 3x}{3x} = 1$ just like $\frac{\sin x}{x}$.

Another way to say it: let $u = 3x$. Then as $x \to 0$, also $u \to 0$, so

$$\lim_{x\to 0} \frac{\sin 3x}{3x} = \lim_{u\to 0} \frac{\sin u}{u} = 1$$

Ex $\lim_{x\to 0} \frac{\sin 6x}{5x}$.

Method 1: $u = 6x$, $\frac{u}{6} = x$ $\quad \lim_{u\to 0} \frac{\sin u}{\left(\frac{5u}{6}\right)} = \frac{6}{5} \lim_{u\to 0} \frac{\sin u}{u} = \frac{6}{5}$

Method 2: $\frac{\sin 6x}{5x} \cdot \frac{6x}{6x} = \frac{\sin 6x}{6x} \cdot \frac{6x}{5x} = \frac{6}{5} \frac{\sin 6x}{6x}$

and $\lim_{x\to 0} \frac{6}{5} \frac{\sin 6x}{6x} = \frac{6}{5} \cdot 1 = \frac{6}{5}$

Chain Rule

How to get the derivative of $F(x) = \sqrt{1+x^2}$?

Think of $F(x)$ as $f \circ g(x)$ i.e. $f(g(x))$ where $g(x) = 1+x^2$
$f(u) = \sqrt{u}$

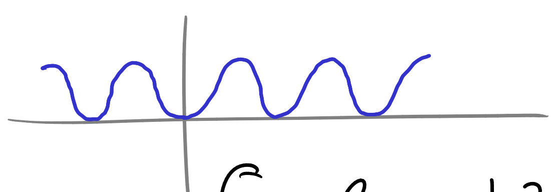
Chain Rule: if g is differentiable at x
and f is differentiable at $g(x)$
and $F = f \circ g$
then $F'(x) = f'(g(x))\, g'(x)$.

Or. if $y = f(u)$, $u = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Ex $y = \sqrt{1+x^2}$ — what is $\frac{dy}{dx}$? say $u = 1+x^2$, $y = \sqrt{u}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}}(2x) = \frac{1}{2\sqrt{1+x^2}} \cdot 2x = \underline{\underline{\frac{x}{\sqrt{1+x^2}}}}$$



Ex $y = \sin^2 x$ say $u = \sin x$, then $y = u^2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot \cos x = 2 \sin x \cos x$$

$= \sin 2x$ — why?
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$!
$\frac{d}{dx}\left(\frac{1}{2}(1-\cos 2x)\right) = \frac{1}{2}(2 \sin 2x)$

(we could also do this by product rule: $y = \sin x \cdot \sin x$ so $\frac{dy}{dx} = \cos x \cdot \sin x + \sin x \cdot \cos x$
$= 2 \sin x \cos x$)

Shorthand: $y = (\sin x)^2$

$$\frac{dy}{dx} = \underbrace{2 \sin x}_{\text{usual power rule formula}} \cdot \cos x$$

derivative of "inside part" $\sin x$

Ex $y = \sin(x^2)$

$$\frac{dy}{dx} = \cos(x^2) \cdot 2x$$

derivative of sin is cos

derivative of "inside part" x^2

Or, longhand: let $u = x^2$, then $y = \sin u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= (\cos u) \cdot (2x)$$

$$= \underline{\cos(x^2) \cdot (2x)}$$

Ex $y = e^{4x}$

$$\frac{dy}{dx} = e^{4x} \cdot \frac{d}{dx}(4x) = 4 \cdot e^{4x}$$

(and in general $\frac{d}{dx} e^{ax} = a e^{ax}$)

Ex $\frac{d}{dx}(2^x) = \frac{d}{dx}\left((e^{\ln 2})^x\right)$

$2 = e^{\log_e 2} = e^{\ln 2}$

$$= \frac{d}{dx}\left(e^{(\ln 2) \cdot x}\right)$$

$$= (\ln 2) \cdot e^{(\ln 2) \cdot x}$$

$$= (\ln 2) \cdot 2^x$$

(and similarly $\frac{d}{dx}(a^x) = (\ln a)\cdot a^x$)

Ex $y = (x^2-3)^{170}$

$$\frac{dy}{dx} = 170\cdot(x^2-3)^{169}\cdot(2x)$$
$$= 340x\,(x^2-3)^{169}$$

$$\frac{d^2y}{dx^2} = 340\left((x^2-3)^{169} + x\cdot 169(x^2-3)^{168}\cdot 2x\right)$$
$$= 340(x^2-3)^{168}(x^2-3+338x^2)$$
$$= 340(x^2-3)^{168}(339x^2-3)$$

Ex $y = \dfrac{1}{\sqrt[4]{x^3+1}}$ $\qquad -\frac{3}{4}x^2\cdot(x^3+1)^{-5/4}$

$$\frac{dy}{dx} = \frac{d}{dx}\left[(x^3+1)^{-1/4}\right] = -\frac{1}{4}(x^3+1)^{-5/4}\cdot 3x^2$$
$$= -\frac{3}{4}x^2(x^3+1)^{-5/4}$$
$$= -\frac{3x^2}{4\sqrt[4]{(x^3+1)^5}}$$

Ex $y = \left(\dfrac{t-2}{2t+1}\right)^9$ $\qquad \dfrac{dy}{dt} = 9\left(\dfrac{t-2}{2t+1}\right)^8 \dfrac{d}{dt}\left(\dfrac{t-2}{2t+1}\right)$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{(2t+1)(1)-(t-2)(2)}{(2t+1)^2}$$
$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{2t+1-2t+4}{(2t+1)^2}$$
$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{5}{(2t+1)^2} = 45\frac{(t-2)^8}{(2t+1)^{10}}$$

Ex $y = e^{\sin x}$

$$\frac{dy}{dx} = e^{\sin x} \cdot \cos x$$

derivative of e^u is e^u deriv. of "inside" $\sin x$

Ex $y = e^{\cos 2x}$

$$\frac{dy}{dx} = e^{\cos 2x} \cdot \frac{d}{dx}(\cos 2x)$$

$$= e^{\cos 2x} \cdot (-\sin 2x) \cdot 2$$

$$= -2 \sin 2x \, e^{\cos 2x}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$\left.\begin{aligned} y &= e^u \\ u &= \cos v \\ v &= 2x \end{aligned}\right\} \rightarrow \frac{dy}{dx} = e^u \cdot (-\sin v) \cdot 2$$

$$= e^{\cos 2x}(-\sin 2x) \cdot 2$$

Why is the Chain Rule true?

Roughly we're looking at a function of the shape $y(u(x))$.

Imagine that x changes a little bit: by some small Δx (from x to $x + \Delta x$)

Then $u(x)$ changes a little: by some small Δu (u is differentiable)

And $y(u(x))$ changes a little, by some small Δy.

What we want is $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. Write $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \left(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\right) \cdot \left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

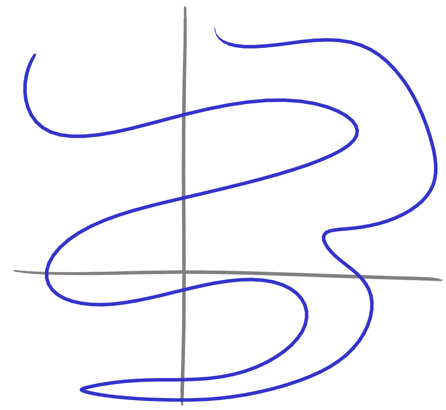
Implicit Differentiation

So far we looked at graphs of functions $y(x)$ usually given by some definite formula,

like $y = e^{x\cos x}$

Sometimes we know a relation between y and x but not a formula for y.

Ex $\quad y^7 + 8yx^3 + 17y^2x^8 = 0 \quad (*)$



Even without a formula for y, we can still find the slope of the tangent line at some given point (x,y) on the graph!

How? Imagine locally there is a function $y(x)$ giving this graph. Then apply $\frac{d}{dx}$ to our equation $(*)$:

$$\frac{d}{dx}\left(y^7 + 8yx^3 + 17y^2x^8\right) = \frac{d}{dx}(0) = 0$$

$$7y^6\frac{dy}{dx} + 8\frac{dy}{dx}\cdot x^3 + 24yx^2 + 17\cdot\left(2y\frac{dy}{dx}x^8 + y^2\cdot 8x^7\right) = 0$$

$$7y^6y' + 8y'x^3 + 24yx^2 + 34yx^8y' + 136y^2x^7 = 0$$

$$y'(7y^6 + 8x^3 + 34yx^8) = -24yx^2 - 136y^2x^7$$

$$y' = -\frac{24yx^2 + 136y^2x^7}{7y^6 + 8x^3 + 34yx^8}$$