Lecture 9

24 Sep 2015

Midterm 1 next Tue in class

Last time: trig limits and trig derivatives

Ex $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Ex $\lim_{x \to 0} \frac{\sin 3x}{3x} = 1$ — why? Plug in e.g. $x = 0.001$ — get $\frac{\sin(0.003)}{0.003}$

$= \frac{\sin(\text{small } \#)}{\text{same small } \#}$

and we know that as the small # goes to 0, $\frac{\sin(\text{small})}{\text{small}}$ goes to 1.

Or: introduce new variable $u = 3x$

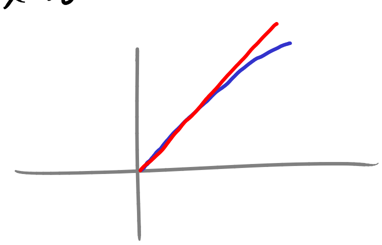
then $\lim_{x \to 0} \frac{\sin 3x}{3x} = \lim_{u \to 0} \frac{\sin u}{u} = 1.$

(and similarly, $\lim_{x \to 0} \frac{\sin kx}{kx} = 1$ for any real # $k \neq 0$.)

Ex $\lim_{x \to 0} \frac{\sin 6x}{5x}$

Method 1: $u = 6x$, $x = \frac{u}{6}$, then $\lim_{u \to 0} \frac{\sin u}{5 \cdot \frac{u}{6}} = \lim_{u \to 0} \frac{6}{5} \cdot \frac{\sin u}{u} = \underline{\underline{\frac{6}{5}}}$

Method 2: $\lim_{x \to 0} \frac{\sin 6x}{5x} \cdot \frac{6x}{6x} = \lim_{x \to 0} \frac{\sin 6x}{6x} \cdot \frac{6x}{5x} = \frac{6}{5} \cdot \lim_{x \to 0} \frac{\sin 6x}{6x} = \underline{\underline{\frac{6}{5}}}$

(Moral: near $x = 0$, $\sin x \approx x$  so $\frac{\sin 6x}{5x} \approx \frac{6x}{5x} = \frac{6}{5}$)

Chain Rule

How to compute derivative of $F(x) = \sqrt{1+x^2}$?

Think of $F(x)$ as $f \circ g(x)$ i.e. $f(g(x))$ where $g(x) = 1+x^2$, $f(u) = \sqrt{u}$

Chain Rule: if g is differentiable at x
and f is differentiable at $g(x)$
then $F = f \circ g$ is differentiable at x

and $F'(x) = f'(g(x))\, g'(x)$

Or: if $y = f(u)$ $u = g(x)$

then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Ex $y = \sqrt{1+x^2}$ say $u = 1+x^2$, $y = \sqrt{u}$

then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \frac{1}{2\sqrt{u}} \cdot 2x$

$= \frac{1}{2\sqrt{1+x^2}} \cdot 2x$

$= \frac{x}{\sqrt{1+x^2}}$

Ex $y = \sin^2 x$ $u = \sin x$ $y = u^2$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= 2u \cdot \cos x = 2 \sin x \cos x$

Remark this is also $= \sin 2x$! Why? Because $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

so $\frac{d}{dx}\sin^2 x = -\frac{1}{2}\frac{d}{dx}\cos 2x$

$= -\frac{1}{2}(-\sin 2x)\cdot\frac{d}{dx}(2x)$

$= \frac{1}{2}\sin 2x \cdot 2$

$= \underline{\sin 2x}$

Ex $\quad y = (\sin x)^n \qquad u = \sin x \quad y = u^n$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= nu^{n-1}\cdot\cos x$$

$$= n(\sin x)^{n-1}\cos x$$

Shorthand: $\quad y = (\sin x)^n$

$$\frac{dy}{dx} = \underbrace{n(\sin x)^{n-1}}_{\text{from power rule}}\cdot\underbrace{\cos x}_{\text{derivative of the "inside function" } \sin x}$$

Ex $\quad y = \sin(x^2) \qquad u = x^2,\ y = \sin(u)$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \cos(u)\cdot(2x)$$

$$= \underbrace{\cos(x^2)}_{\text{deriv. of sine function}}\cdot\underbrace{(2x)}_{\text{deriv. of "inside" function } x^2}$$

Ex $\quad y = e^{4x}$

$$\frac{dy}{dx} = e^{4x}\cdot\frac{d}{dx}(4x) = e^{4x}\cdot 4 = \underline{4e^{4x}}$$

(and similarly, $\frac{d}{dx}e^{ax} = ae^{ax}$)

Remark: if $a=1$, this says $\frac{d}{dx} e^x = 1 \cdot e^x$ ✓

Ex $y = 2^x$

$\frac{dy}{dx} = ?$

$$y = 2^x = \left(e^{\log_e 2}\right)^x$$

(Notation: "ln" = "$\log_e$")

$$= \left(e^{\ln 2}\right)^x$$

$$= e^{(\ln 2)\cdot x}$$

so $\frac{dy}{dx} = (\ln 2)\cdot e^{(\ln 2)\cdot x} = (\ln 2)\cdot 2^x$

$$\frac{d}{dx}\left(2^x\right) = (\ln 2)\cdot 2^x$$

Ex $y = (x^2+4)^{130}$

$$\frac{dy}{dx} = 130\,(x^2+4)^{129}\cdot 2x$$

$$= 260x\,(x^2+4)^{129}$$

Ex $y = \left(\frac{t-2}{2t+1}\right)^9$

$$\frac{dy}{dx} = 9\left(\frac{t-2}{2t+1}\right)^8 \frac{d}{dt}\left(\frac{t-2}{2t+1}\right)$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{(2t+1)(1)-(t-2)(2)}{(2t+1)^2}$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{2t+1-2t+4}{(2t+1)^2}$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{5}{(2t+1)^2} = 45\cdot\frac{(t-2)^8}{(2t+1)^{10}}$$

Ex $y = e^{\sin x}$

$$\frac{dy}{dx} = e^{\sin x} \cdot \cos x$$

(or, longhand: $y = e^u$ $u = \sin x$ $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot \cos x = e^{\sin x} \cdot \cos x$)

Ex $y = e^{(e^{(e^x)})}$

$$\frac{dy}{dx} = e^{(e^{(e^x)})} \cdot \frac{d}{dx}\left(e^{(e^x)}\right)$$

$$= e^{(e^{(e^x)})} \cdot e^{(e^x)} \cdot \frac{d}{dx}(e^x)$$

$$= e^{(e^{(e^x)})} \cdot e^{(e^x)} \cdot e^x$$

$$= e^{\left(e^{(e^x)} + e^x + x\right)}$$

Ex $\frac{d}{dx}\left(e^{\cos 2x}\right) = e^{\cos 2x} \cdot \frac{d}{dx}(\cos 2x)$

$$= e^{\cos 2x} \cdot (-\sin 2x) \, 2$$

$$= -2(\sin 2x)\, e^{\cos 2x}$$

Why is Chain Rule true?

Have a function $y(u(x))$

Make a small change in x, say Δx $(x \rightsquigarrow x + \Delta x)$

Then $u(x)$ changes by a small amount, Δu $(\Delta u = u(x+\Delta x) - u(x))$

Then $y(u(x))$ changes by a small amount, Δy $(\Delta y = y(u+\Delta u) - y(u))$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

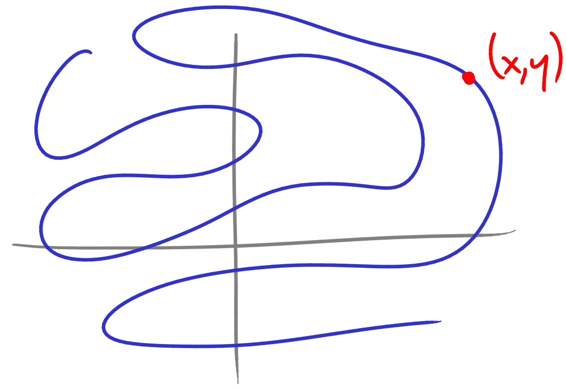
i.e. $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Implicit Differentiation

We know how to find slope of tangent line to a graph given as $y = f(x)$.

How about a relation like:

$$y + y^2 x^3 + y^3 x^2 + x^{11} = 0 \quad (*)$$



Near a given point (x,y) there is a function $y(x)$ giving this graph. (not vertical tangent)

Now apply $\frac{d}{dx}$ to both sides of $(*)$:

$$\frac{dy}{dx} + \left(2y\frac{dy}{dx} \cdot x^3 + y^2 \cdot 3x^2\right) + \left(3y^2 \frac{dy}{dx} \cdot x^2 + y^3 \cdot 2x\right) + 11x^{10} = 0$$

$$\left(1 + 2yx^3 + 3y^2x^2\right)\frac{dy}{dx} + y^2 3x^2 + y^3 2x + 11x^{10} = 0$$

$$\frac{dy}{dx} = -\frac{3y^2x^2 + 2xy^3 + 11x^{10}}{1 + 2yx^3 + 3y^2x^2}$$