Lecture 12

8 Oct 2015

HW06 due Sat 3am
HW07 due Wed 3am

Regarding HW06:

(A) $\frac{d}{dx} \ln\left(\frac{(x+4)^3}{(x-2)^2}\right)$

Hard way: chain rule —

$$\frac{1}{\left(\frac{(x+4)^3}{(x-2)^2}\right)} \frac{d}{dx}\left(\frac{(x+4)^3}{(x-2)^2}\right)$$

Easy way: $\ln\left(\frac{(x+4)^3}{(x-2)^2}\right) = \ln\left((x+4)^3\right) - \ln\left((x-2)^2\right)$

$$= 3\ln(x+4) - 2\ln(x-2)$$

so $\frac{d}{dx} \ln\left(\frac{(x+4)^3}{(x-2)^2}\right) = 3\frac{d}{dx}\ln(x+4) - 2\frac{d}{dx}\ln(x-2)$

$$= \underline{\underline{\frac{3}{x+4} - \frac{2}{x-2}}}$$

(B) $\frac{d}{dx}\left(\frac{x}{4}\right) = \frac{d}{dx}\left(\frac{1}{4}\cdot x\right) = \frac{1}{4}$

(this is much easier than doing quotient rule!)

(C) How to simplify $\sin(\tan^{-1}x)$? i.e. if $\theta = \tan^{-1}x$ what is $\sin\theta$?

i.e. if $\tan\theta = x$ what is $\sin\theta$?

Algebraic way: find trig identities that relate $\tan\theta$ to $\sin\theta$

$$\tan^2\theta = x^2$$

$$\cot^2\theta = \frac{1}{x^2}$$

$$1+\cot^2\theta = 1+\frac{1}{x^2}$$

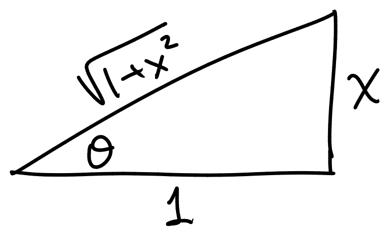
$$\csc^2\theta = 1 + \frac{1}{x^2}$$

$$\frac{1}{\csc^2\theta} = \frac{1}{1+\frac{1}{x^2}}$$

$$\sin^2\theta = \frac{1}{1+\frac{1}{x^2}}$$

$$\sin\theta = \frac{1}{\sqrt{1+\frac{1}{x^2}}}\cdot\frac{x}{x} = \frac{x}{\sqrt{1+\frac{1}{x^2}}\cdot\sqrt{x^2}} = \frac{x}{\sqrt{1+x^2}}$$

Simpler way: $\tan\theta = x$ what is $\sin\theta$?



$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = x$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{\sqrt{1+x^2}}$$

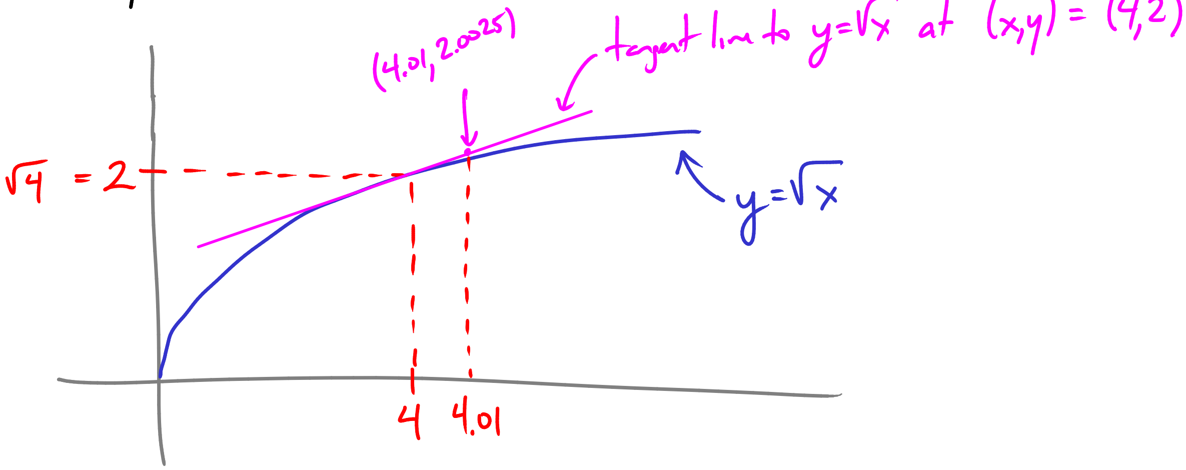
Today: another use of derivatives —

Linear Approximation

What is $\sqrt{4}$? $\sqrt{4} = 2$

What is $\sqrt{4.01}$? "a little bigger than 2"

Let's try to estimate:



Tangent line: $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$. At $x=4$: $\frac{dy}{dx} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$ — slope of tangent line

Tangent line is line thru $(4,2)$ with slope $\frac{1}{4}$: $y-2 = \frac{1}{4}(x-4)$

$y = 2 + \frac{1}{4}(x-4)$

Plug in $x = 4.01$: $y = 2 + \frac{1}{4}(4.01-4)$

$= 2 + \frac{1}{4}(0.01) = 2.0025$

So: 2.0025 is our estimate of $\sqrt{4.01}$

and in fact $\sqrt{4.01} = 2.002498\ldots$

Similarly: estimate $\sqrt{4.03}$. Tangent line at $(3,4)$ is $y = 2 + \frac{1}{4}(x-4)$

plug in $x = 4.03$: get $y = 2 + \frac{1}{4}(4.03-4)$

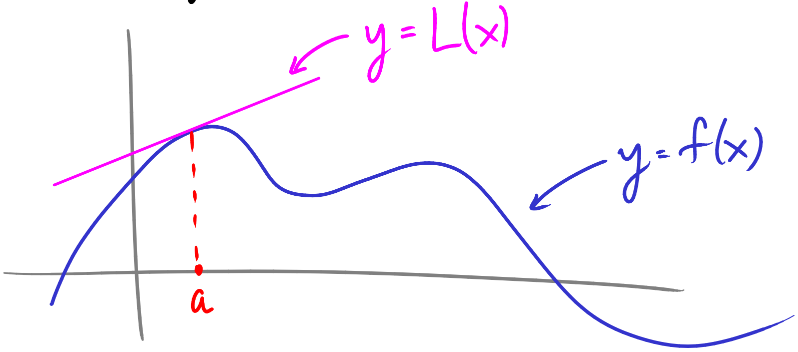
$= 2 + \frac{1}{4}(0.03) = 2.0075$

So 2.0075 is our estimate of $\sqrt{4.03}$

and $\sqrt{4.03} = 2.007486\ldots$

x	$2+\frac{1}{4}(x-4)$	$\sqrt{x}$
4	2	2
4.01	2.0025	2.002498...
4.03	2.0075	2.007486...
6	2.5	2.4495...
8	3	2.8284...
3.99	1.9975	1.997...

Generally:



$y=L(x)$ is the tangent line to $y=f(x)$ at $(a,f(a))$.

i.e. $y=L(x)$ is the line thru $(a,f(a))$ with slope $f'(a)=m$

i.e. $y-f(a)=m(x-a)$

$$L(x)-f(a)=f'(a)(x-a)$$

$$L(x)=f(a)+f'(a)(x-a)$$

Call $L(x)$ the linear approximation to $f(x)$ at a.

<u>Ex</u> Estimate $\cos\left(\frac{\pi}{2}+0.02\right)$.

Write $f(x)=\cos(x)$. Let $a=\frac{\pi}{2}$.

$f(a)=\cos\left(\frac{\pi}{2}\right)=0$

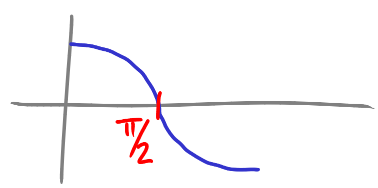
$f'(a)=-\sin\left(\frac{\pi}{2}\right)=-1$.

Linear approx to $f(x)$ at $a=\pi/2$:

$$\begin{aligned} L(x) &= f(a)+f'(a)(x-a) \\ &= 0+(-1)\left(x-\tfrac{\pi}{2}\right) \\ &= \tfrac{\pi}{2}-x \end{aligned}$$

So $L\left(\frac{\pi}{2}+0.02\right)=\frac{\pi}{2}-\left(\frac{\pi}{2}+0.02\right)=-0.02$

So, our estimate is $\cos\left(\frac{\pi}{2}+0.02\right)\approx -0.02$



<u>Ex</u> Estimate $\sqrt[3]{25}$.

We'll do this by linear approx of $f(x)=\sqrt[3]{x}$, <u>at the point</u> $a=27$.

$$L(x)=f(a)+f'(a)(x-a)$$

$f(a)=\sqrt[3]{27}=3$

$f'(x)=\frac{d}{dx}\left(x^{1/3}\right)=\frac{1}{3}x^{-2/3}$

$f'(a)=\frac{1}{3}(27)^{-2/3}$

$$f(a) = 3$$
$$f'(a) = \frac{1}{27}$$

$$= \frac{1}{3}\left(\left(27\right)^{\frac{1}{3}}\right)^{-2}$$
$$= \frac{1}{3}(3)^{-2}$$
$$= \frac{1}{3}\left(\frac{1}{9}\right) = \frac{1}{27}$$

so $L(x) = 3 + \frac{1}{27}(x-27)$

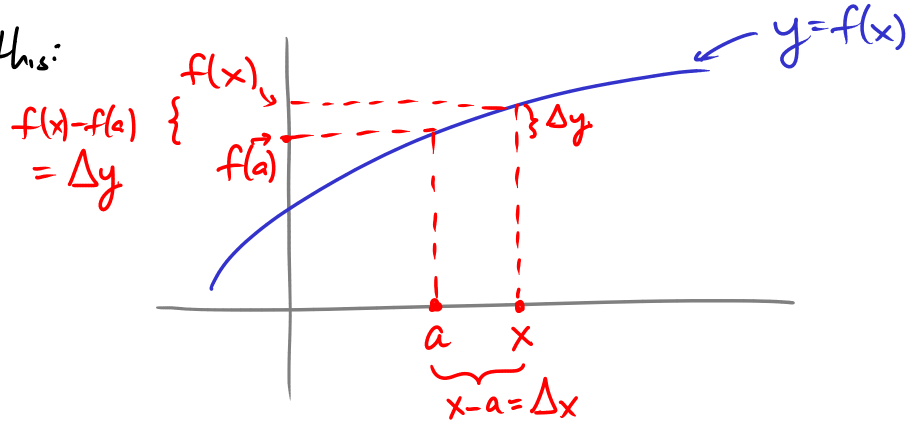
Our estimate: $f(25) \approx L(25) = 3 + \frac{1}{27}(25-27)$
$$= 3 - \frac{2}{27} = \frac{79}{27}.$$

So $\sqrt[3]{25} \approx \frac{79}{27}$.

Similarly for $\sqrt[3]{26}$: $L(26) = 3 + \frac{1}{27}(26-27)$
$$= 3 - \frac{1}{27} = \frac{80}{27}.$$

for $\sqrt[3]{28}$: $L(28) = 3 + \frac{1}{27} = \frac{82}{27}.$

Another way to think about this:



$$f(x) = f(a) + \Delta y$$
$$= f(a) + \left(\frac{\Delta y}{\Delta x}\right)\Delta x$$

If Δx is very small then $\frac{\Delta y}{\Delta x}$ is about $\frac{dy}{dx}$

$$\left(\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\right)$$

so $f(x) \approx f(a) + \left(\frac{dy}{dx}\right)\Delta x$
$$= f(a) + f'(a) \cdot \Delta x$$

$= f(a) + f'(a) \cdot (x-a)$

Differential of f: (just notation)

write "$df = \frac{df}{dx} \cdot dx$"

as analog of $\Delta f \approx \frac{df}{dx} \Delta x$ for small Δx

think of df as "infinitesimally small" Δf

dx as "infinitesimally small" Δx

e.g. $$d(\tan x) = \frac{d(\tan x)}{dx} \cdot dx = \sec^2 x \, dx$$

$$d(\tan x) = \sec^2 x \, dx$$

so for small Δx, $\Delta(\tan x) \approx \sec^2 x \, \Delta x$

Ex Estimate $\ln(0.98)$. Idea: $0.98 \approx 1$. $\ln(1) = 0$.

Imagine moving from $x=1$ to 0.98: $x = 1$

$\Delta x = 0.98 - 1 = -0.02$

$$d(\ln x) = \frac{d(\ln x)}{dx} dx = \frac{1}{x} dx$$

$$\Delta(\ln x) \approx \frac{1}{x} \Delta x$$

$$= \frac{1}{1} \cdot (-0.02)$$

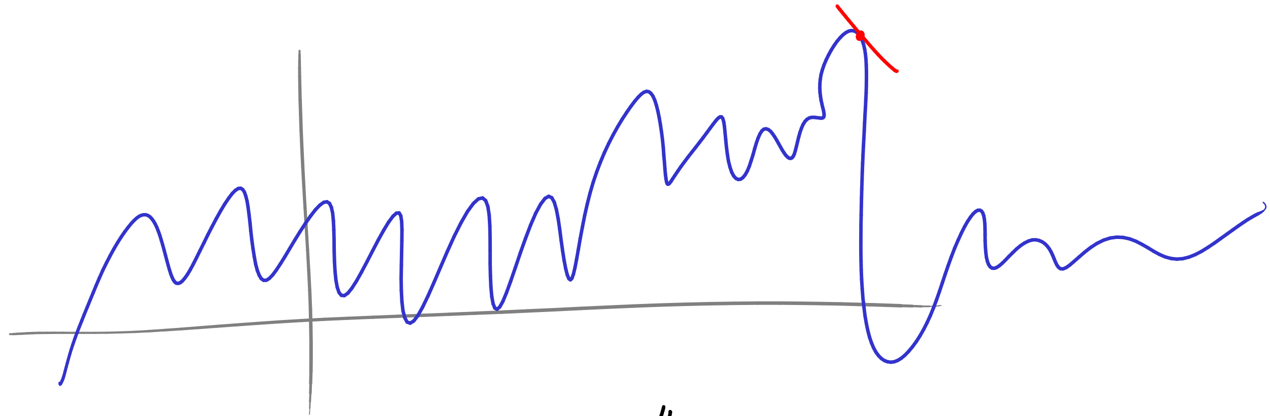
$$= -0.02$$

So $\ln(0.98) = \ln(x + \Delta x) \approx \ln(x) + \Delta(\ln x)$

$$= 0 + (-0.02)$$

$$= -0.02$$

Q: How do we know a priori whether our estimate is reliable?



Need to know estimate of size of $f''(x)$ for x near a.

Ex What is the linear approximation to $f(x) = 3x^5$ at $a=1$?

$f(1) = 3$

$f'(x) = 15x^4$, so $f'(1) = 15$

so $L(x) = f(1) + f'(1)\cdot(x-1)$

$= 3 + 15(x-1)$

so e.g. $f(1.001) \approx 3 + 15(1.001 - 1) = 3 + 15(0.0001) = 3.0015$

$\parallel$

$3(1.001)^5$

Remark: try expanding out $3(1+0.001)^5$

$= 3\cdot(1+0.001)(1+0.001)(1+0.001)(1+0.001)(1+0.001)$

$= 3\left(1 + 5 \times 0.001 + (\text{much smaller terms})\right)$

$\approx 3(1 + 0.0005) = 3(1.0005) = 3.0015$