Lecture 13

13 Oct 2015

Today's office hour: 2-3pm

Last time: linear approximation e.g.

$$\sqrt[3]{28} \approx 3 + \frac{1}{27} \qquad f(x) = \sqrt[3]{x}$$

$$\sqrt[3]{29} \approx 3 + \frac{2}{27} \qquad f(27) = 3$$

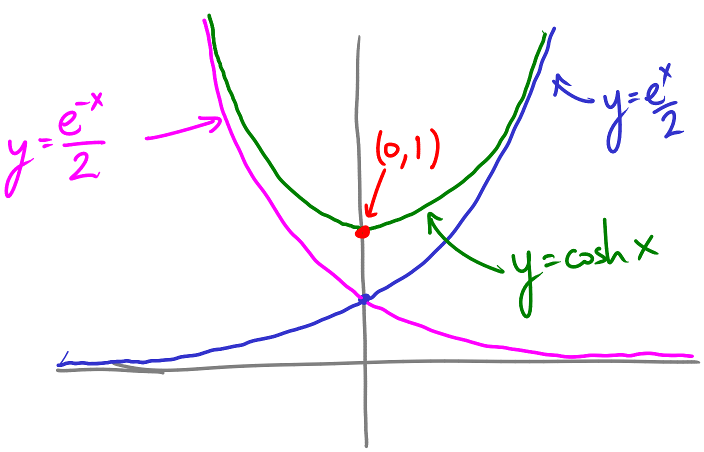
$$\sqrt[3]{31} \approx 3 + \frac{4}{27} \quad \ldots \qquad f'(27) = \frac{1}{3}\frac{1}{x^{2/3}} = \frac{1}{3}\cdot\frac{1}{9} = \frac{1}{27}$$

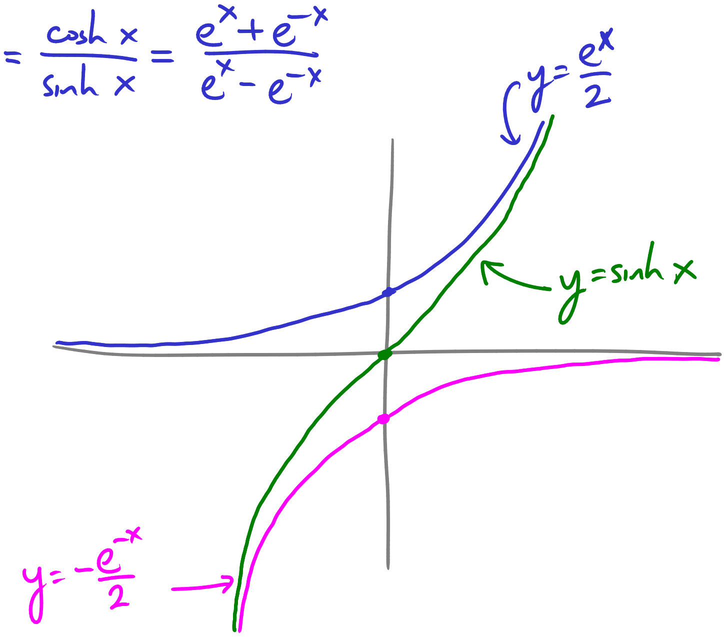
Hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$





$\sinh x$ and $\cosh x$ both have <u>domain</u> $= (-\infty, \infty)$

$\sinh x$ has <u>range</u> $= (-\infty, \infty)$ $\qquad$ $\cosh x$ has <u>range</u> $= [1, \infty)$

Remark $y = a \cdot \cosh\left(\frac{x}{b}\right)$ a, b constants

is the shape of a freely-hanging heavy cord (e.g. power line) ("catenary")

Hyperbolic identities

$\sinh(-x) = -\sinh x$ $\cosh(-x) = \cosh x$

$\cosh^2 x - \sinh^2 x = 1$ $1 - \tanh^2 x = \operatorname{sech}^2 x$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

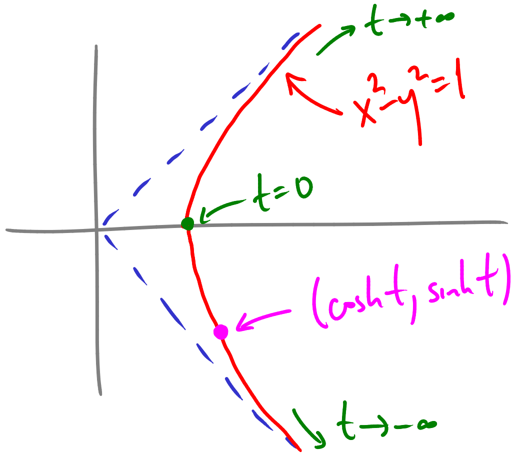
$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Why? e.g. $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

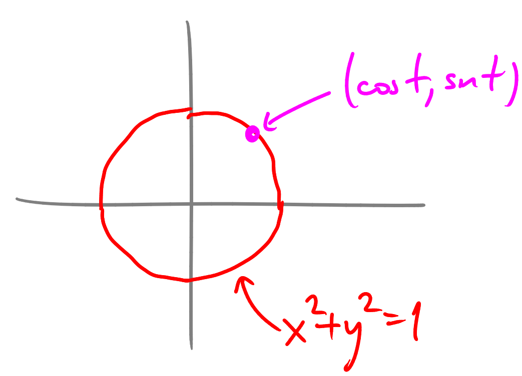
$$\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^{x}}{2} = -\left(\frac{e^x - e^{-x}}{2}\right) = -\sinh x \checkmark$$

e.g.
$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4}$$
$$= \frac{(e^{2x} + 2 \cdot e^x e^{-x} + e^{-2x}) - (e^{2x} - 2 \cdot e^x e^{-x} + e^{-2x})}{4}$$
$$= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{4} = \frac{4}{4} = 1 \checkmark$$

So, $\cosh^2 t - \sinh^2 t = 1$: if we let $x = \cosh t$, $y = \sinh t$ then $x^2 - y^2 = 1$



cf.



Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \leftarrow \left(\text{why? } \frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right) = \frac{e^x-e^{-x}}{2} = \sinh x\right)$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

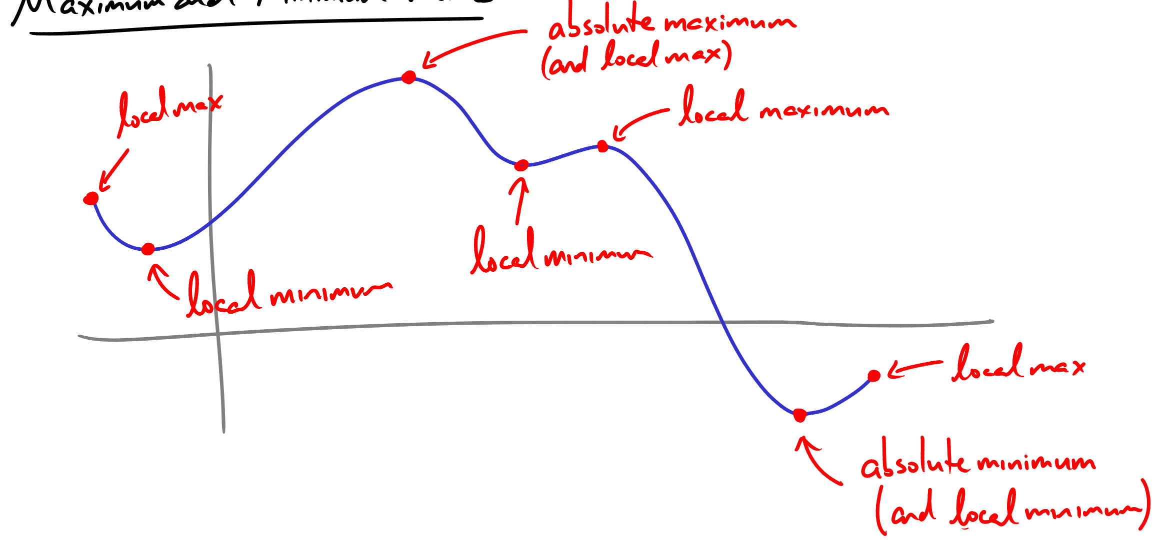
Ex $\frac{d}{dx}(8\cosh 2x) = 16\sinh 2x$

Remark

Similarity to usual trig functions is "explained" by

$$\cos x = \frac{e^{ix}+e^{-ix}}{2} \qquad \sin x = \frac{e^{ix}-e^{-ix}}{2}$$

Maximum and Minimum Values



Say $f(c)$ is <u>absolute max</u> value of f if $f(c) \geq f(x)$ for every x in domain of f.

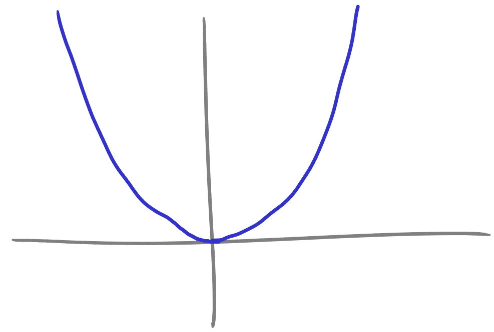
Say $f(c)$ is <u>absolute min</u> value of f if $f(c) \leq f(x)$ " " " " " " ".

Say $f(c)$ is <u>local max</u> value of f if $f(c) \geq f(x)$ for every x <u>near</u> c.
Say $f(c)$ is <u>local min</u> value of f if $f(c) \leq f(x)$ " " " " ".

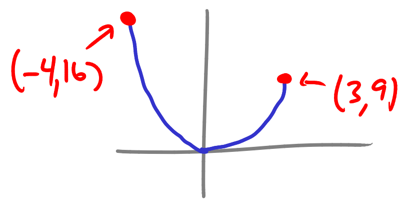
<u>Ex</u> $f(x) = x^2$ domain $(-\infty, \infty)$.

absolute minimum: $f(0) = 0$.
local minimum: $f(0) = 0$
(<u>no others</u>)

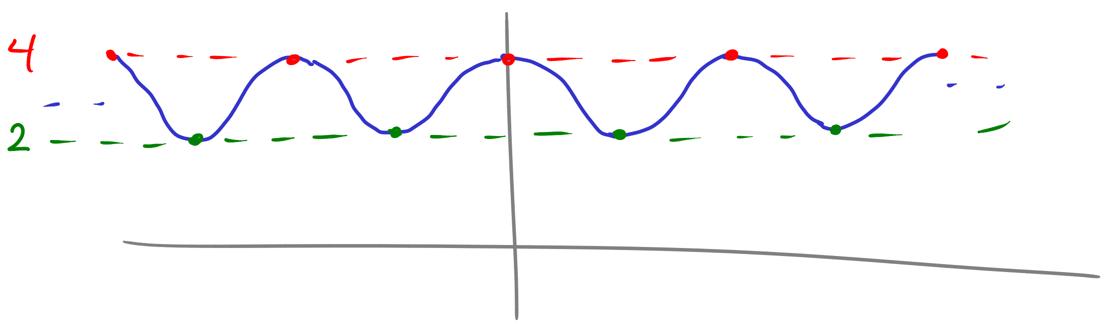
<u>no</u> local or absolute maxima.



(But, if we pick a different domain we may have a max.
e.g. if domain $= [-4, 3]$ then have absolute max at $f(-4) = 16$
local max at $f(-4) = 16$
$f(3) = 9$)

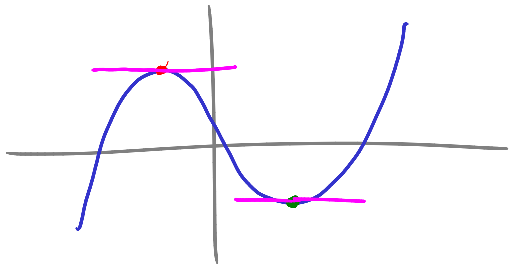


<u>Ex</u> $f(x) = 3 + \cos x$
local and absolute max: $f(x) = 4$ at $x = 0, 2\pi, -2\pi, 4\pi, -4\pi, \ldots$
local and absolute min: $f(x) = 2$ at $x = \pi, -\pi, 3\pi, -3\pi, \ldots$



<u>Fact</u>: If $f(x)$ has a local max or min at $x = c$,
<u>and</u> $f'(c)$ <u>exists</u>,
<u>and</u> the domain of f includes an interval containing c: (i.e. c isn't at the <u>edge</u> of the domain)

<u>Then</u>, $f'(c) = 0$.



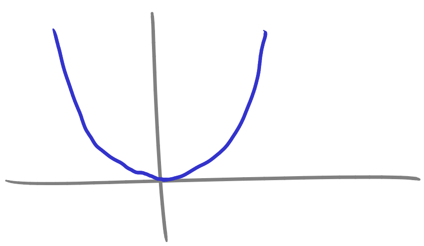
ie: the places where the graph $y=f(x)$ turns around and is differentiable are places where the graph has a horizontal tangent.

Ex $f(x)=x^2$

$f'(x)=2x$, so: ① $f'(x)$ exists for all x
② $f'(x)=0$ only if $x=0$

Domain $=(-\infty,\infty)$

So: the only possible place for a local max/min to occur is at $x=0$



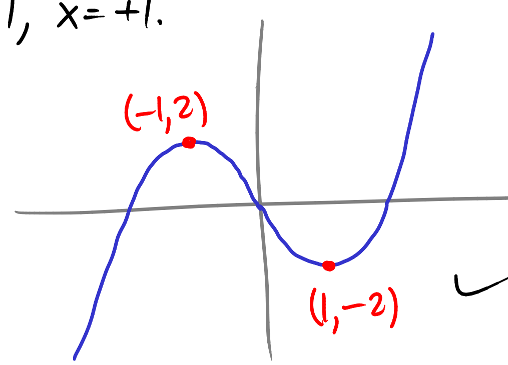
Indeed, we do have a local min at $x=0$, and nowhere else ✓

Ex $f(x)=x^3-3x$ $\quad f'(x)=3x^2-3$ $\quad$ domain $=(-\infty,\infty)$

$f'(x)$ exists for all x,

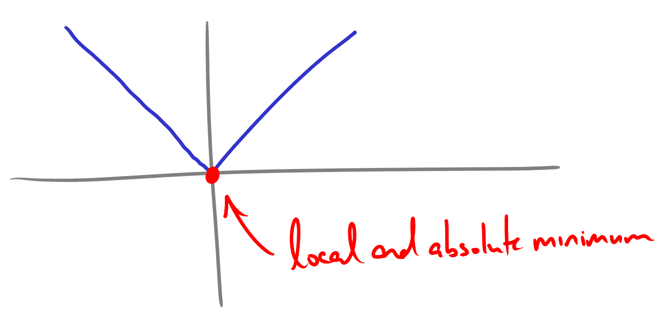
$f'(x)=0$ only at
$$0=3x^2-3$$
$$0=3(x-1)(x+1)$$
$$x=+1,\ x=-1$$

So the only possible local max/min are at $x=-1$, $x=+1$.



✓

Ex $f(x)=|x|$



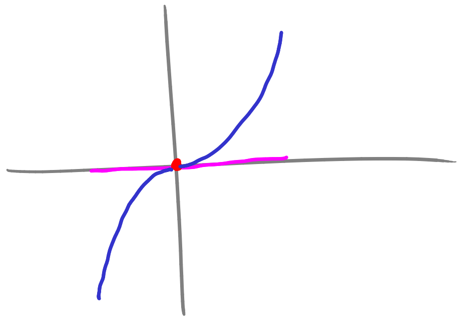
Here $f'(x)=\begin{cases} 1 & \text{at } x>0 \\ -1 & \text{at } x<0 \\ \text{DNE} & \text{at } x=0 \end{cases}$

So $x=0$ is the only possible local max/min. ✓

Ex $f(x) = x^3$ domain $= (-\infty, \infty)$

$f'(x) = 3x^2$ so $f'(x) = 0$ only at $x = 0$.

So the only possible local max/min is at $x = 0$.



But in fact here $x=0$ is not a local max or min.

Thus $f(x)$ has no local max or min.

Strategy for finding absolute max/min for a function f with domain $[a,b]$:

1. find values of f at all "critical numbers": x where $f'(x) = 0$ or $f'(x)$ DNE.
2. find values of $f(a)$, $f(b)$.
3. take max, min values of f from this list.

Ex Find absolute max, min of

$$f(x) = 12 + 4x - x^2 \quad \text{on} \quad [0,5] \qquad (0 \leq x \leq 5)$$

1. no x for which $f'(x)$ DNE.

   $f'(x) = 4 - 2x$ so $f'(x) = 0$ means $4 - 2x = 0$

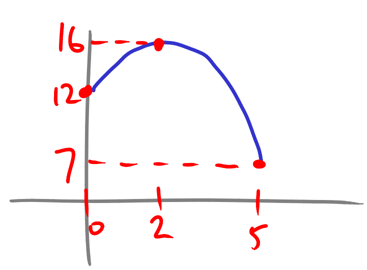
   $x = 2$

   So the only critical # is $x = 2$.

   $f(2) = 12 + 8 - 4 = 16$.

2. $f(0) = 12 + 0 - 0 = 12$.

   $f(5) = 12 + 20 - 25 = 7$.



3. absolute max is $f(2) = 16$.

   absolute min is $f(5) = 7$.