Lecture 14

15 Oct 2015

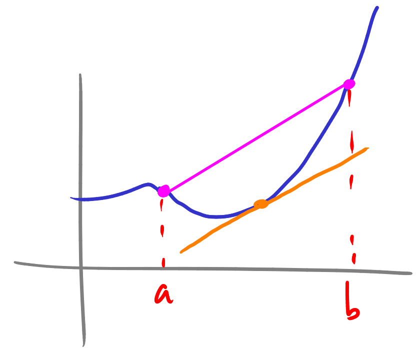
Last time: maxima and minima of functions

Mean Value Theorem

Fact: Suppose f is a function continuous on $[a,b]$
differentiable on (a,b)

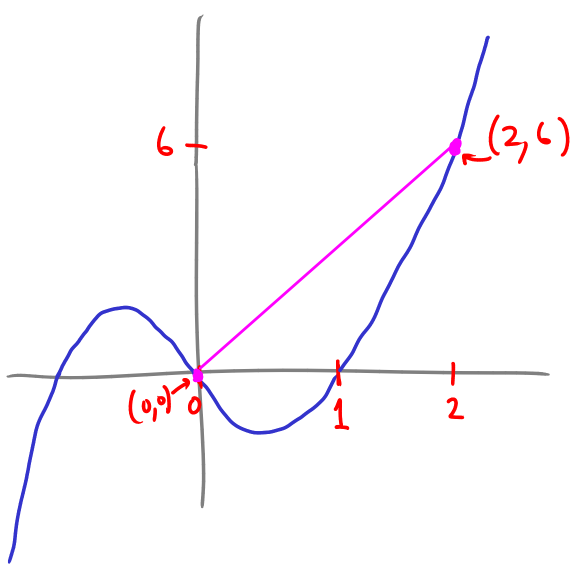
Then there is some c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



Ie, there is some c in (a,b) where the slope of the tangent line to the graph $y=f(x)$ at $(c,f(c))$ is equal to slope of the secant line connecting $(a,f(a))$ and $(b,f(b))$.

Ex $f(x) = x^3 - x$



slope of secant line is $\frac{6-0}{2-0} = 3$

MVT says there must be some c in $(0,2)$ such that $f'(c)=3$

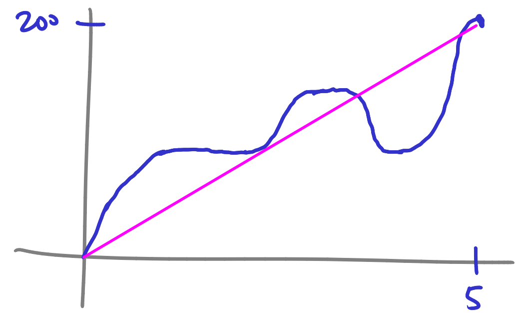
Let's check: $f'(x) = 3x^2-1$

so $f'(c)=3$ means $3c^2-1=3$

$3c^2=4 \quad c^2=\frac{4}{3} \quad c=\frac{2}{\sqrt{3}} > 1$ ← ? looks funny!

Ex Suppose we drive 200 miles in 5 hours.
Let the position be $x(t)$.

$x(0) = 0$
$x(5) = 200$



Slope of secant line $= \frac{200}{5} = 40$ miles/hour
(= average speed)

MVT $\Longrightarrow$ at some moment, the speedometer read exactly 40 mph.

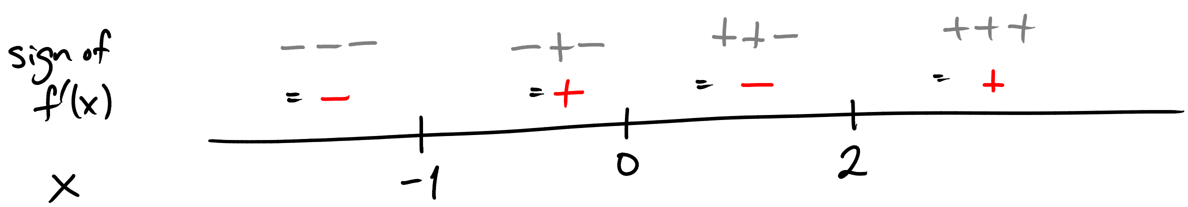
Graphing using derivatives

How do we use $f'(x)$ to get information about the graph of $f(x)$?

Ex Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$
is increasing and where it is decreasing.

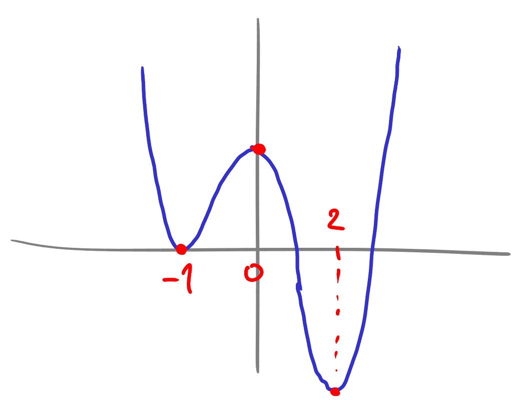
$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$= 12x(x^2 - x - 2)$$
$$= 12x(x+1)(x-2)$$

To see whether $f'(x)$ is +ve or −ve, look at these 3 pieces:



So $f(x)$ is increasing for x in $(-1, 0) \cup (2, \infty)$
$f(x)$ is decreasing for x in $(-\infty, -1) \cup (0, 2)$

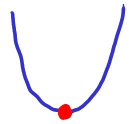
$f(-1) = 0$
$f(0) = 5$
$f(2) = -27$



Let's look closer at the critical points.
$f'(x) = 0$ at $x = -1, 0, 2$.

At $x=2$:

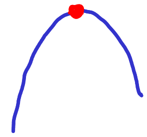
sign of $f'(x)$	$-$	$+$
x	2	



local minimum

At $x=0$:

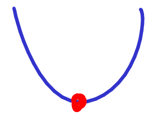
$+$	$-$
0	



local maximum

At $x=-1$:

$-$	$+$
-1	



local minimum

First Derivative Test

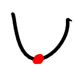
If c is a critical number for $f(x)$,

① If $f'(x)$ changes sign from +ve to −ve at c then f has local max at c.

$+ \mid -$

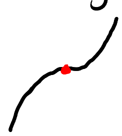
② If $f'(x)$ " " " −ve to +ve " " " " " local min at c.

$- \mid +$

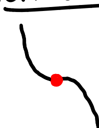


③ If $f'(x)$ does not change sign at c then f has neither max nor min at c.

$+ \mid +$



$- \mid -$



Ex Find all local max/min of $f(x) = x^{1/3}(x+4)$ on $(0,\infty)$ and $(-\infty,0)$

Find critical numbers:

$$f(x) = x^{4/3} + 4x^{1/3}$$

$$f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}\left(x^{1/3} + x^{-2/3}\right)$$

$$= \frac{4}{3}x^{-2/3}(x+1)$$

$f'(x)=0$ only at $x=-1$.

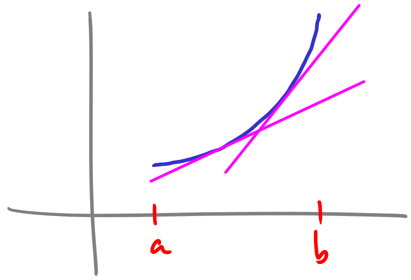
⟶ on $(0,\infty)$: no local max/min

on $(-\infty,0)$:

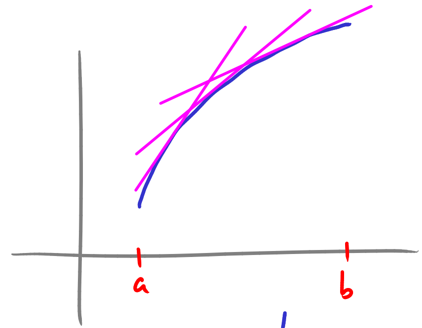
$+-$	$++$	
$-$	$+$	
	-1	0

So $x=-1$ is a local minimum.

Concavity



concave up



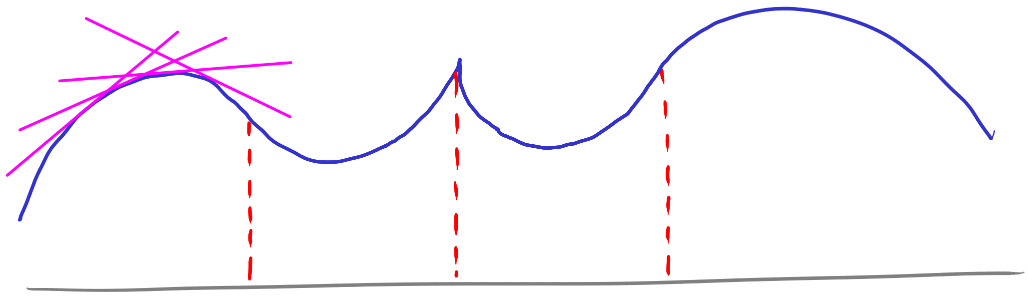
concave down

Both of these have $f'(x) > 0$ for x in (a,b)

but they are different:

Say graph of $y = f(x)$ is **concave up** on (a,b) if it lies above all of its tangent lines in (a,b).

Say graph of $y = f(x)$ is **concave down** on (a,b) if it lies below all of its tangent lines in (a,b).



Fact

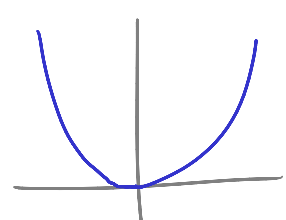
If $f''(x) < 0$ for all x in (a,b) then the graph of f is concave down on (a,b).

If $f''(x) > 0$ for all x in (a,b) then the graph of f is concave up on (a,b).

Ex $f(x) = x^2$

$f'(x) = 2x$

$f''(x) = 2$ so $f''(x) > 0$ for all x in $(-\infty, \infty)$

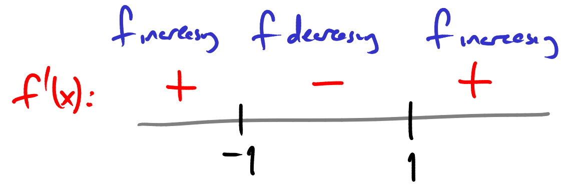


so the graph $y = x^2$ is concave up for all x in $(-\infty, \infty)$.

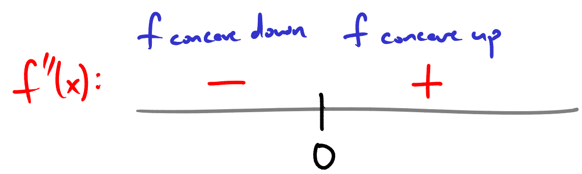
A <u>point of inflection</u> is a point $(c, f(c))$ where f is continuous and the graph $y = f(x)$ <u>changes</u> from concave up to concave down or vice versa.

<u>Ex</u> $f(x) = x^3 - 3x$

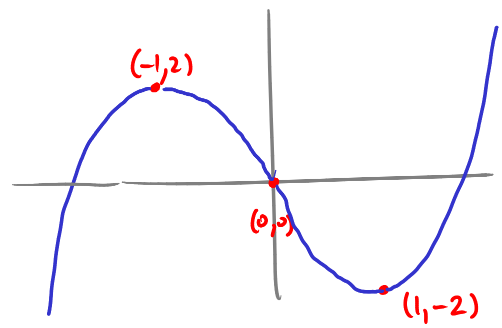
$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$
$= 3(x+1)(x-1)$



$f''(x) = 6x$



So: $x = 1$ is local minimum
$x = 0$ is inflection point
$x = -1$ is local maximum



<u>Second Derivative Test</u>

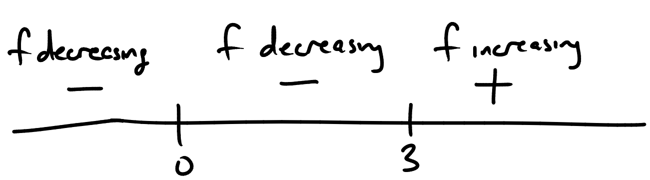
If f is continuous at c, $f'(c) = 0$, and

1. $f''(c) > 0$, <u>then</u> c is <u>local minimum</u>
2. $f''(c) < 0$, <u>then</u> c is <u>local maximum</u>
3. $f''(c) = 0$, then the test fails — gives no information

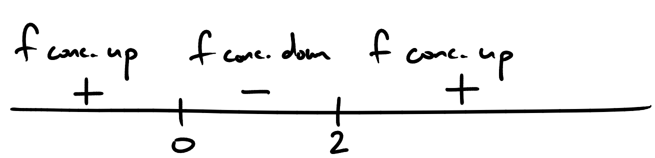
<u>Ex</u> Sketch $y = x^4 - 4x^3$.

$f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$
$= 4x^2(x-3)$



$f''(x) = 12x^2 - 24x$
$= 12x(x-2)$



critical pts: $x = 0$ neither max nor min
$x = 3$ local min

inflection pts: $x = 0$, $x = 2$

$f(0) = 0$
$f(2) = 16 - 32 = -16$
$f(3) = 81 - 108 = -27$