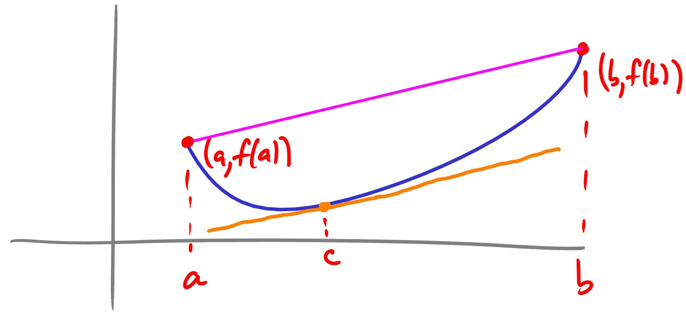
# Lecture 14

15 Oct 2015

Last time: minima and maxima

## Mean Value Theorem




**Fact:** Say $f$ is continuous on $[a,b]$
differentiable on $(a,b)$

Then, there is some $c$ in the interval $(a,b)$
such that $$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

**Ex** $f(x) = x^3 - x$

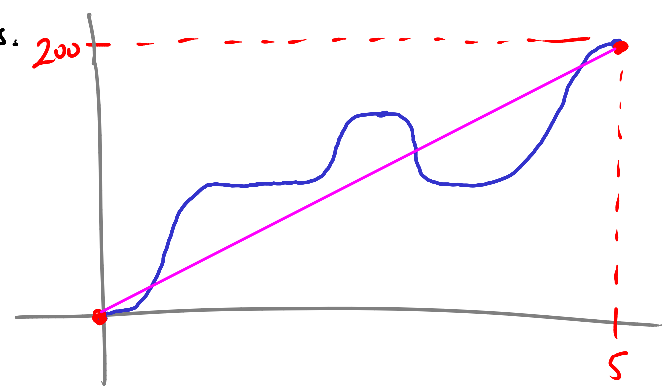
$a=0 \quad f(a)=0$
$b=2 \quad f(b) = 8-2=6$

MVT says: there is some $c$ between 0 and 2
such that $f'(c) = \frac{f(b)-f(a)}{2-0} = \frac{6-0}{2-0} = 3$

To check: $f'(x) = 3x^2 - 1$

want $f'(c) = 3$ i.e. $3c^2 - 1 = 3$
$3c^2 = 4$
$c^2 = 4/3$
$c = \pm \frac{2}{\sqrt{3}}$. Indeed $c = \frac{2}{\sqrt{3}}$ is between 0 and 2 ✓

**Ex** Suppose we drive 200 miles in 5 hours.
Let $x(t)$ be our position at time $t$.




$x(0) = 0$
$x(5) = 200$

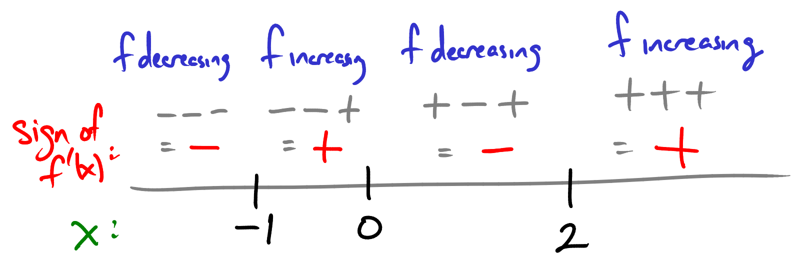
$$\frac{x(5)-x(0)}{5-0} = \frac{200}{5} = 40 \text{ miles/hr}$$
(average velocity)

MVT says: there's some $c$ for which $x'(c) = 40$
ie there's some moment when speedometer reads exactly 40mph.

# Graphing Using Derivatives

**Ex** Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$= 12x(x^2 - x - 2)$$
$$= 12x(x-2)(x+1)$$



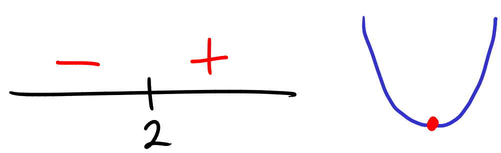

[ Plug in $x > 2$: this is $12 \cdot (\text{positive}) \cdot (\text{positive}) \cdot (\text{positive})$
so it's positive
Similarly work out signs in all other intervals ]

So: $f$ is increasing if $x$ is in $(-1, 0) \cup (2, \infty)$

$f$ is decreasing if $x$ is in $(0, 2) \cup (-\infty, -1)$

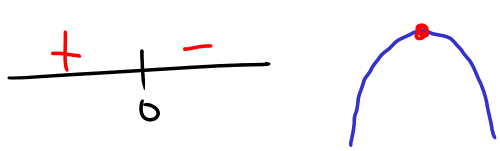
Let's look closer at the critical points: $f'(x) = 0$ at $-1, 0, 2$

At $x = 2$, sign of $f'(x)$



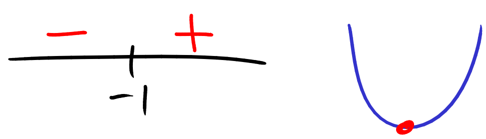

so $f(2)$ is local minimum

At $x = 0$,

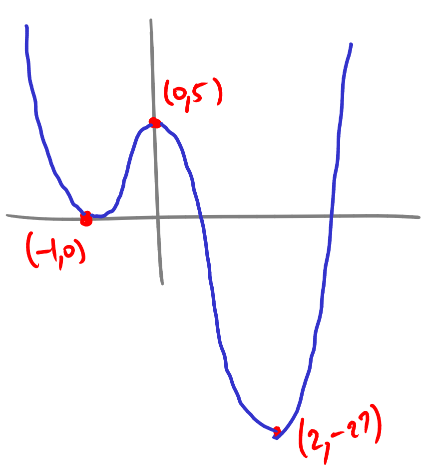



so $f(0)$ is local max

At $x = -1$,



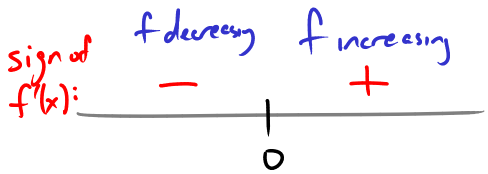

so $f(-1)$ is local min

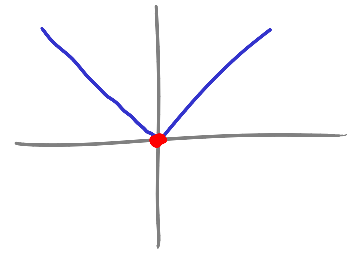



$f(-1) = 0$
$f(0) = 5$
$f(2) = -27$

Ex $f(x) = |x|$

$$f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ \text{DNE} & x = 0 \end{cases}$$



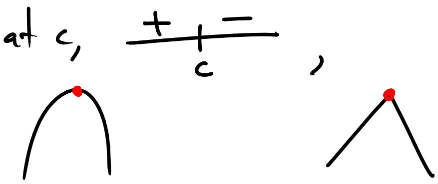
only critical point is $x = 0$, local minimum

Formalizing the idea we just used:

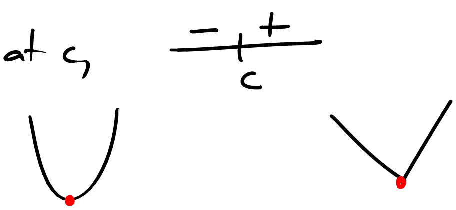
First Derivative Test

If $c$ is a critical point of $f$, and $f$ is continuous at $c$,

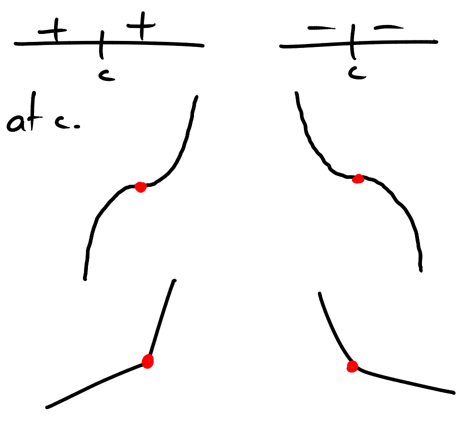
① If $f'(x)$ changes from positive to negative at $c$,
then $f(x)$ has local maximum at $c$.




② If $f'(x)$ changes from negative to positive at $c$,
then $f(x)$ has local minimum at $c$.




③ If $f'(x)$ does not change sign at $c$,
then $f(x)$ has neither min nor max at $c$.




Ex Find all local min/max of $f(x) = x^{1/3}(x+4)$ for $x$ in $(-\infty, 0)$.

First step: find all critical pts.

$$f(x) = x^{4/3} + 4x^{1/3}$$

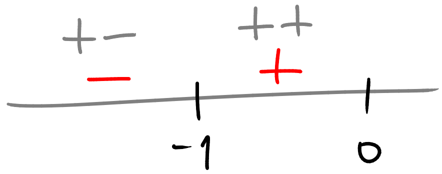
$$f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}x^{-2/3}(1+x)$$

$f'(x) = 0$ only at $x=-1$. And $f'(x)$ exists for all $x$ in $(-\infty, 0)$.

$\rightarrow$ $x=-1$ is the only critical point in $(-\infty, 0)$.

sign of $f'(x)$:



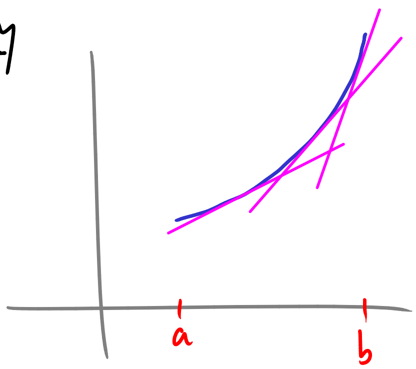

$x^{-2/3}$: is $(x^{-1/3})^2$ – always positive

$(1+x)$: is positive if $x>-1$, is negative if $x<-1$

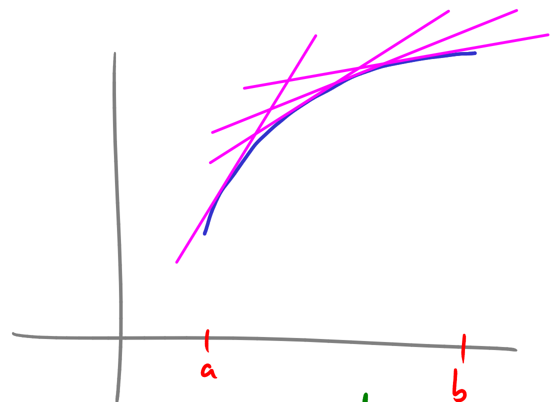
So, $x=-1$ is a local minimum

and there is no local maximum

# Concavity




concave up

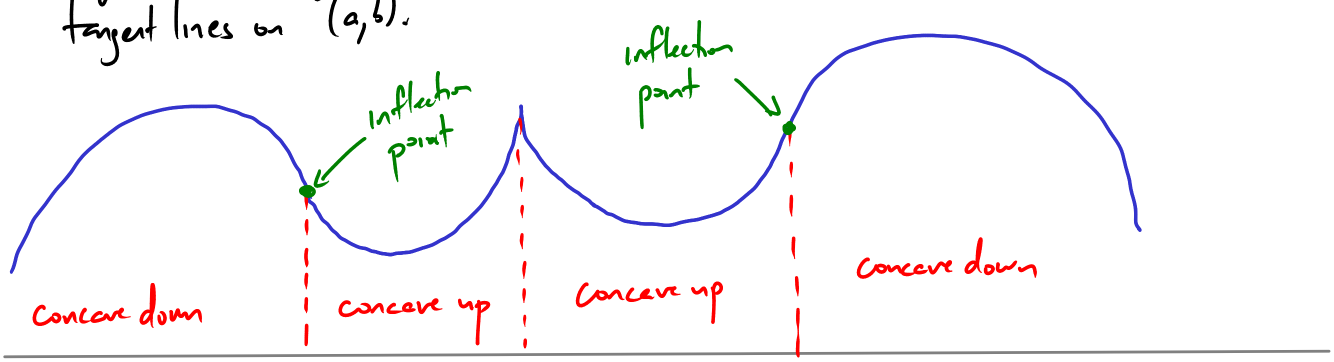



concave down

Both of these functions are increasing, ie $f'(x)>0$ for all $x$ in $(a,b)$. But they are different:

We say the graph $y=f(x)$ is concave up on $(a,b)$ if it lies above all of its tangent lines on $(a,b)$.

We say the graph $y=f(x)$ is concave down on $(a,b)$ if it lies below all of its tangent lines on $(a,b)$.

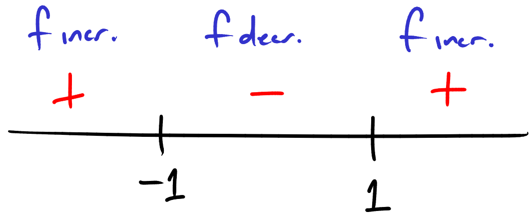



We say a point $(c, f(c))$ is an inflection point of the graph $y=f(x)$ if the concavity changes from up to down or vice versa at $(c, f(c))$.

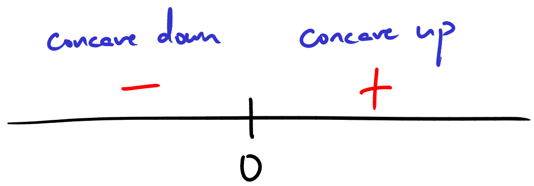
**Fact** ① If $f''(x)>0$ for all $x$ in $(a,b)$ then $f$ is <u>concave up</u> on $(a,b)$.
② If $f''(x)<0$ " " " " " " " " <u>concave down</u> on $(a,b)$.

**Ex** $f(x) = x^3 - 3x$

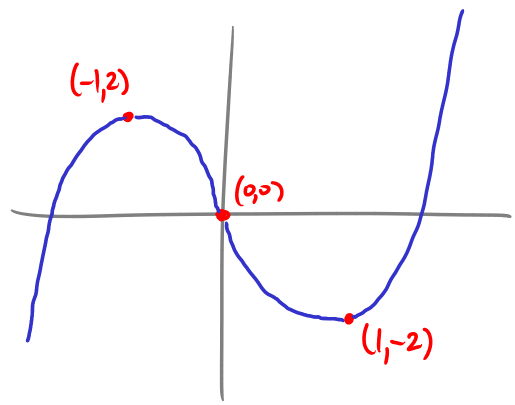
$$f'(x) = 3x^2 - 3$$
$$= 3(x^2-1)$$
$$= 3(x+1)(x-1)$$




$$f''(x) = 6x$$

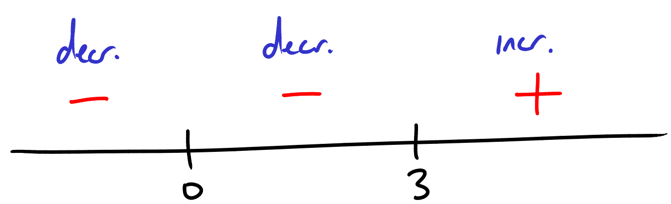



$-1$ is a <u>local max</u> ∩ $\quad f(-1) = 2$
$+1$ is a <u>local min</u> ∪ $\quad f(+1) = -2$
$0$ is <u>inflection point</u> $\quad f(0) = 0$

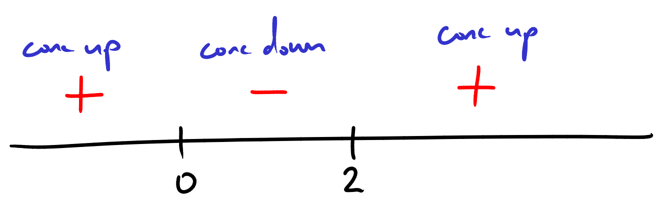



**Ex** Sketch graph of $f(x) = x^4 - 4x^3$.

$$f'(x) = 4x^3 - 12x^2$$
$$= 4x^2(x-3)$$

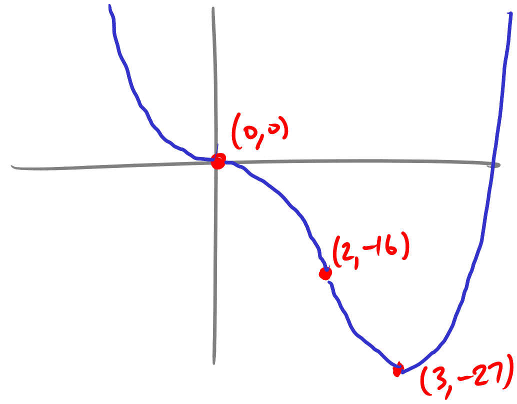



$$f''(x) = 12x^2 - 24x$$
$$= 12x(x-2)$$




crit pts: $0$ neither min or max $\qquad$ inflection pts: $0, 2$
$\qquad\quad 3$ local min

$f(0)=0 \qquad f(2)=-16 \qquad f(3)=-27$




## Second Derivative Test

If $c$ is a crit pt of $f(x)$ and

1. $f''(c) > 0$, then $c$ is local min. $\cup$
2. $f''(c) < 0$, then $c$ is local max. $\cap$
3. $f''(c) = 0$ or $f''(c)$ DNE,
   then test fails — no information.