Lecture 18

3 Nov 2015

Exam 2 average $\approx 81\%$

Office hr today 4–5:30 RLM 9.134

HW due Fri 3am

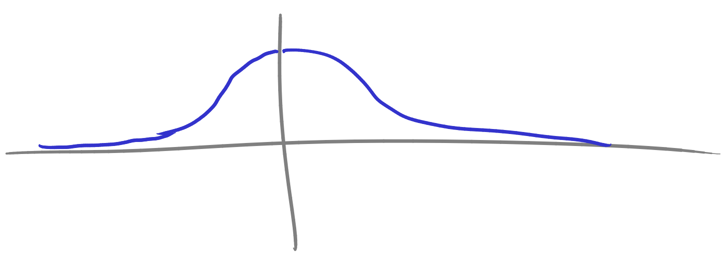
Last time: antiderivatives

Ex $f(x) = x^4 + x$ has antiderivative $\frac{x^5}{5} + \frac{x^2}{2} + C$ for any constant C.

Rk Every continuous function $f(x)$ has an antiderivative

But e.g. the antiderivative of $f(x) = e^{-x^2}$

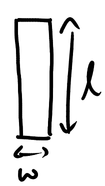
cannot be written in terms of "elementary" functions $(+, \times, -, \div, \exp, \log, \sin, \cos, \ldots)$



So we give it a new name: "error function"

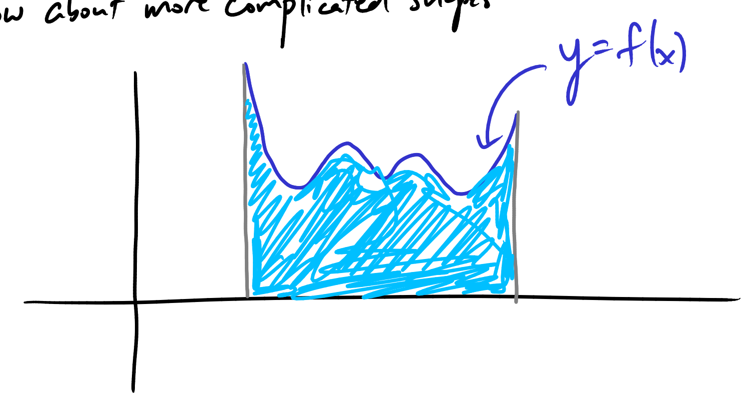
Areas (Ch 5.1)

We all know the area of simple shapes



$A = l \cdot w$

How about more complicated shapes?



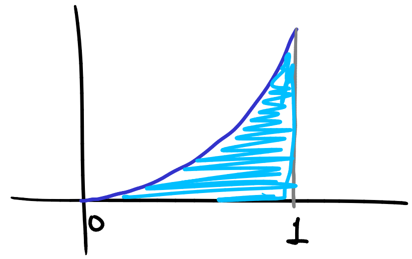
Could try to estimate area by counting boxes.

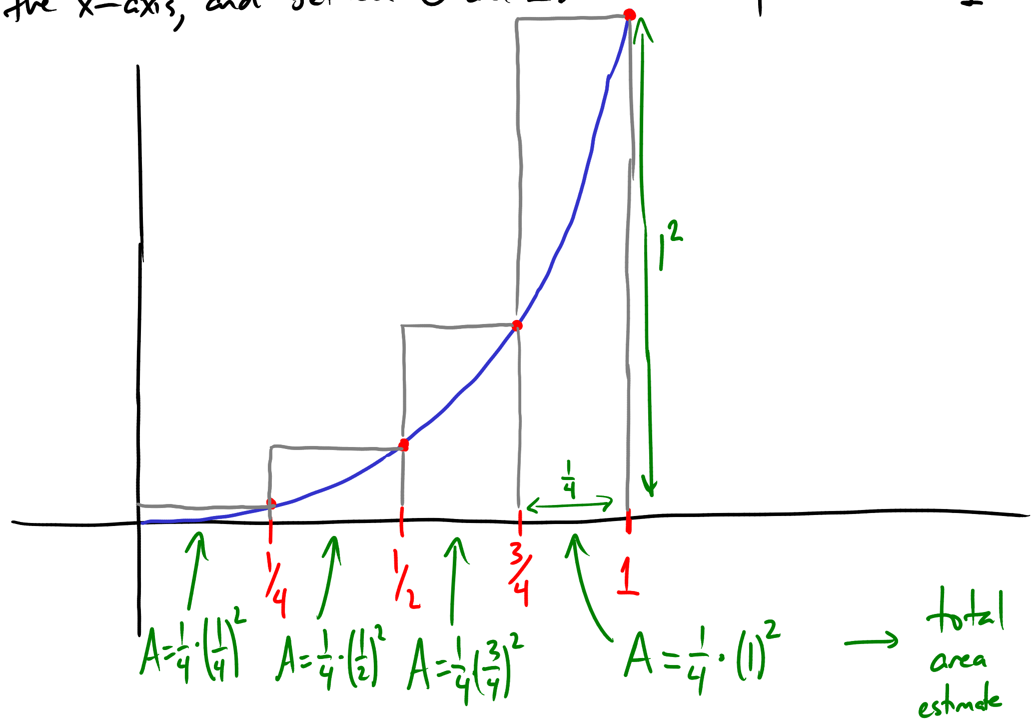
Not exact, but if grid squares very small, it's close.

(Get exact answer by taking a limit as the size of squares $\to 0$.)

Ex Say $f(x)=x^2$.

Estimate the area of the region between $y=f(x)$ and the x-axis, and between 0 and 1.





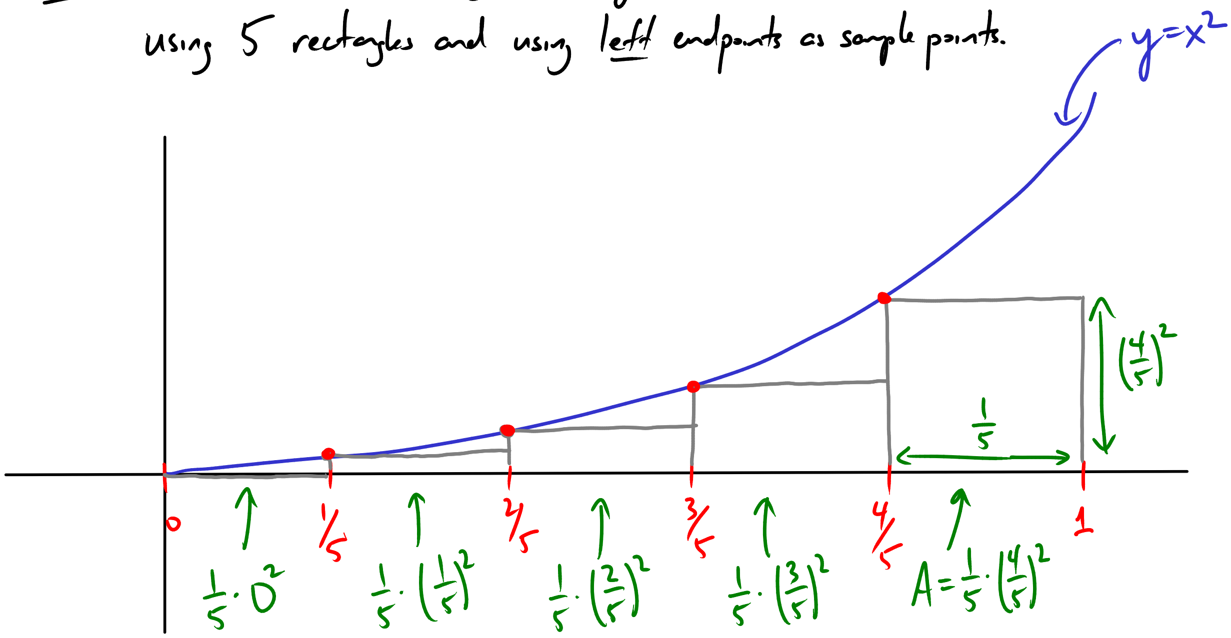
$\rightarrow$ total area estimate $=\frac{1}{4}\left[\left(\frac{1}{4}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{3}{4}\right)^2+(1)^2\right]$

$=\frac{15}{32}$

This is an overestimate of the actual area.

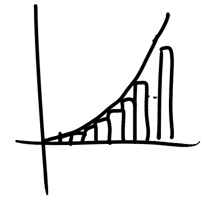
It is the "estimated area using 4 rectangles and using right endpoints as our sample points." So call it R_4: $R_4=\frac{15}{32}$.

Ex Estimate area between graph of $y=x^2$ and x-axis, and between $x=0$ and $x=1$, using 5 rectangles and using left endpoints as sample points.



Estimated area $L_5 = \frac{1}{5}\cdot\left(0^2+\left(\frac{1}{5}\right)^2+\left(\frac{2}{5}\right)^2+\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2\right)$

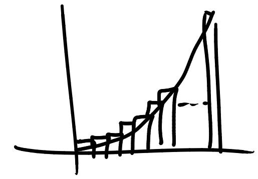
This is an underestimate of the actual area.



Now suppose we used 100 rectangles. Then we would get

$$L_{100} = \frac{1}{100}\cdot\left(0^2+\left(\frac{1}{100}\right)^2+\left(\frac{2}{100}\right)^2+\cdots+\left(\frac{99}{100}\right)^2\right) = 0.328350$$

$$R_{100} = \frac{1}{100}\cdot\left(\left(\frac{1}{100}\right)^2+\left(\frac{2}{100}\right)^2+\left(\frac{3}{100}\right)^2+\cdots+(1)^2\right) = 0.338350$$

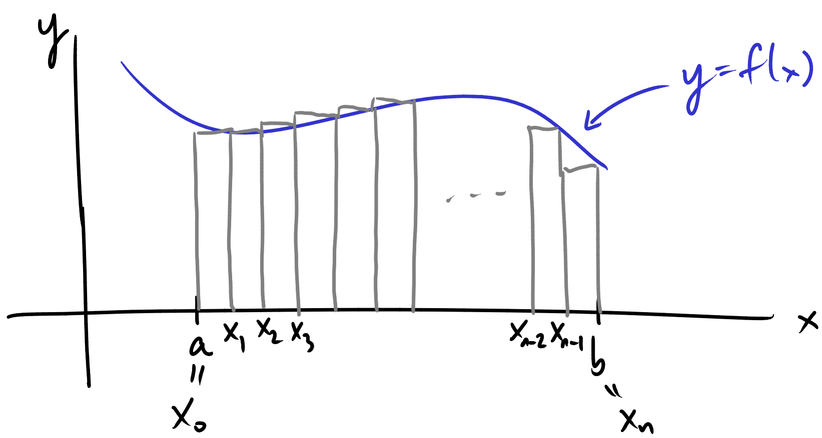


n	L_n	R_n
10	.285	.385
100	.3285	.3385
1000	.33285	.33385

As $n\to\infty$, both L_n and R_n approach $\frac{1}{3}$. (e.g. $R_n = \frac{(n+1)(2n+1)}{6n^2}$)

So, $\frac{1}{3}$ is the exact area under the graph $y=x^2$ between $x=0$ and $x=1$.

For a general $f(x)$, estimate the area similarly:



Width of each rectangle: $\Delta x = \frac{b-a}{n}$

Heights of rectangles: (using right endpoints) $f(x_1), f(x_2), f(x_3), \ldots, f(x_n)$

where $x_i = x_0 + i\Delta x$
$= a + i\Delta x$

$\longrightarrow$ estimated area $R_n = \Delta x\left(f(x_1)+f(x_2)+f(x_3)+\cdots+f(x_n)\right)$

Another convenient notation: ("sigma notation")

The symbol $\sum_{i=1}^{n} f(x_i)$ means $f(x_1)+f(x_2)+\cdots+f(x_n)$.

Ex What is $\sum_{i=1}^{4} i^2$?

$$\sum_{i=1}^{4} i^2 = 1^2+2^2+3^2+4^2 = \underline{\underline{30}}.$$

Ex Write $\frac{2^3}{n}+\frac{4^3}{n}+\frac{6^3}{n}+\cdots+\frac{(2n)^3}{n}$ in sigma notation.

$$\frac{2^3}{n}+\frac{4^3}{n}+\cdots+\frac{(2n)^3}{n} = \sum_{i=1}^{n} \frac{(2i)^3}{n}.$$

In this notation, $R_n = \Delta x \sum_{i=1}^{n} f(x_i)$

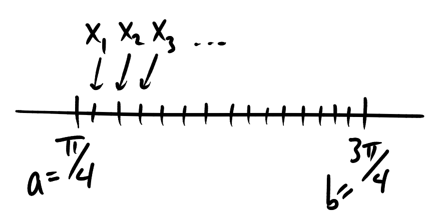
and similarly, $L_n = \Delta x \sum_{i=1}^{n} f(x_{i-1})$

The $\underline{\text{actual}}$ area is $A = \lim_{n\to\infty} R_n$ $\underline{\text{or}}$ $A = \lim_{n\to\infty} L_n$ (both are the same!)

Ex Let A be the area of the region under the graph of $f(x)=\sin^2 x$ between $x=\frac{\pi}{4}$ and $x=\frac{3\pi}{4}$. Using right endpoints as sample points,

- Write a formula for A as a limit.

$$a=\pi/4 \quad b=\frac{3\pi}{4} \quad \Delta x = \frac{b-a}{n} = \frac{\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)}{n} = \frac{\pi}{2n}$$



$$x_i = a + i\Delta x = \frac{\pi}{4} + i\cdot\frac{\pi}{2n}$$

$$R_n = \Delta x \sum_{i=1}^{n} f(x_i) = \frac{\pi}{2n}\cdot \sum_{i=1}^{n} \sin^2(x_i)$$

$$= \frac{\pi}{2n}\cdot \sum_{i=1}^{n} \sin^2\left(\frac{\pi}{4}+i\frac{\pi}{2n}\right)$$

$$A = \lim_{n\to\infty} R_n = \underline{\underline{\lim_{n\to\infty}\left[\frac{\pi}{2n}\sum_{i=1}^{n} \sin^2\left(\frac{\pi}{4}+i\frac{\pi}{2n}\right)\right]}}$$

- Estimate A using 3 rectangles.

$$R_3 = \frac{\pi}{6} \cdot \sum_{i=1}^{3} \sin^2\left(\frac{\pi}{4} + i\frac{\pi}{6}\right)$$

$$= \frac{\pi}{6}\left(\sin^2\left(\frac{\pi}{4}+\frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{4}+\frac{2\pi}{6}\right) + \sin^2\left(\frac{\pi}{4}+\frac{3\pi}{6}\right)\right)$$

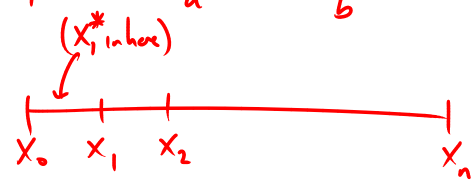
$$\approx 1.23885$$

Definite integrals

Say $f(x)$ is a function defined for $a \le x \le b$.

Divide $[a,b]$ into n equal subintervals of width Δx, endpoints $\underset{a}{x_0}, x_1, \ldots, \underset{b}{x_n}$

Pick any "sample points" x_i^* in $[x_{i-1}, x_i]$.



The definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i^*)\cdot\Delta x}_{\text{"Riemann sums"}}$$

if that limit exists!

(It always does, if f is continuous.)

Ex Write the definition of $\displaystyle\int_1^3 \sqrt{x}\,dx$.

Divide the interval $[1,3]$ into n equal subintervals with width $\Delta x = \frac{3-1}{n} = \frac{2}{n}$

endpoints $x_0 = 1,\ x_1 = 1+\Delta x,\ x_2 = 1+2\Delta x, \ldots$

$$x_i = 1 + i\Delta x = 1 + \frac{2i}{n}$$

Choose right endpoints: $x_i^* = x_i$

Then $$\int_1^3 \sqrt{x}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} \Delta x \cdot f(x_i^*) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2}{n}\right)\cdot\sqrt{1+\frac{2i}{n}}$$