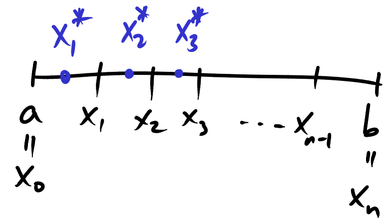
Lecture 19

Last time: definition of <u>definite integral</u>:

Given a function f on $[a,b]$

divide $[a,b]$ into n equal subintervals, width $\Delta x = \frac{b-a}{n}$



$$x_i = x_0 + i\,\Delta x = x_0 + i\,\frac{b-a}{n}$$

Pick sample points x_i^* in the interval $[x_{i-1}, x_i]$

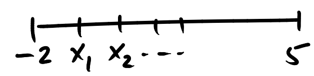
e.g. <u>left endpoint</u>: $x_i^* = x_{i-1}$

<u>right endpoint</u>: $x_i^* = x_i$

Then $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

<u>Ex</u> Write the definition of $\int_{-2}^{5} e^{2x+1}\,dx$ as a limit, using left endpoints.

$[-2,5]$ $\quad a=-2 \quad b=5 \quad \Delta x = \frac{b-a}{n} = \frac{5-(-2)}{n} = \frac{7}{n}$



$$x_i = a + i\,\Delta x$$
$$= -2 + i\frac{7}{n}$$

Left endpoints: $x_i^* = x_{i-1} = -2 + (i-1)\frac{7}{n}$ $\qquad f(x) = e^{2x+1}$

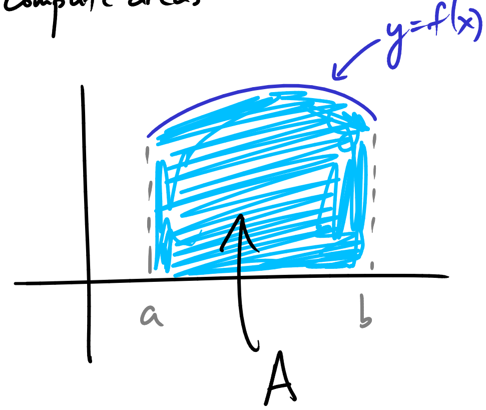
Then $$\int_a^b f(x)\,dx = \int_{-2}^{5} e^{2x+1}\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-2 + (i-1)\tfrac{7}{n}\right)\cdot\frac{7}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} e^{2\left(-2+(i-1)\frac{7}{n}\right)+1}\cdot\frac{7}{n}$$

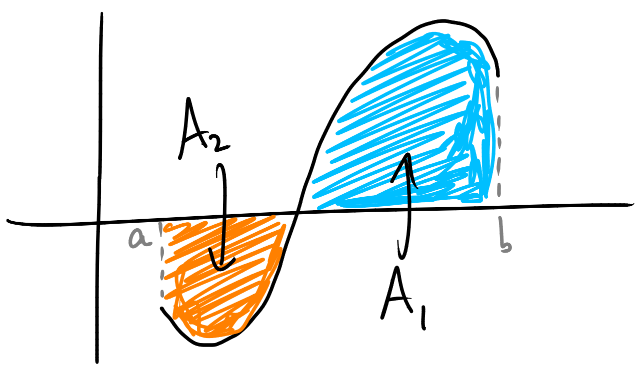
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(e^{-3+14\frac{(i-1)}{n}}\right)\frac{7}{n}$$

Facts about integrals

Integrals compute areas:



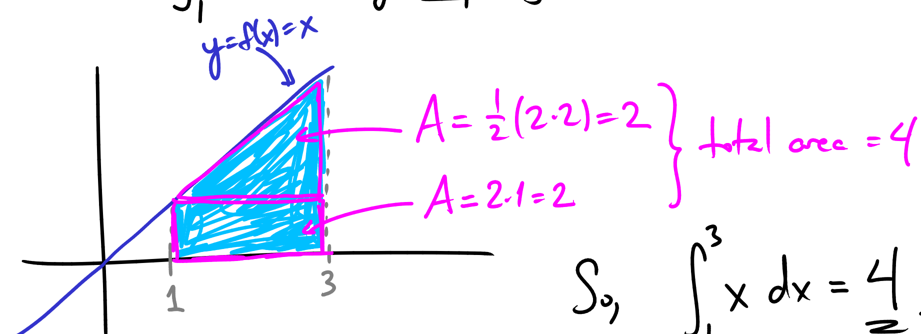
$$\int_a^b f(x)\,dx = A$$



$$\int_a^b f(x)\,dx = A_1 - A_2$$

$$\int_a^b |f(x)|\,dx = A_1 + A_2$$

Ex Evaluate $\int_1^3 x\,dx$ by interpreting it as an area.



So, $\int_1^3 x\,dx = \underline{\underline{4}}$.

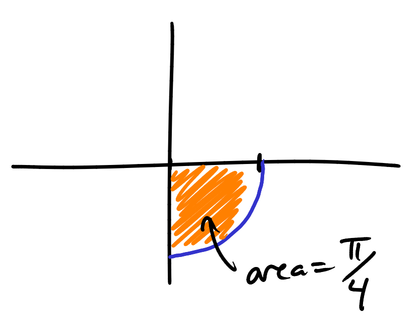
Ex Evaluate $\int_0^1 -\sqrt{1-x^2}\,dx$ by interpreting it as an area.

We're looking at $y = -\sqrt{1-x^2}$

so $y^2 = 1 - x^2$

$y^2 + x^2 = 1 \longrightarrow$ part of the unit circle

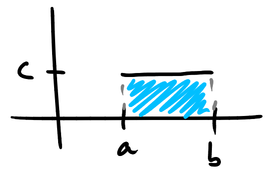
x goes from 0 to 1



So $$\int_0^1 -\sqrt{1-x^2}\,dx = -\frac{\pi}{4}$$

minus because the region is below the x-axis!

Integral rules:



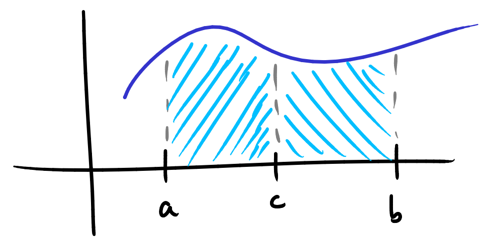
1) $\int_a^b c\,dx = c(b-a)$

2) $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3) $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$

4) $\int_a^b f(x)-g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

5) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



So far we always considered $\int_a^b f(x)\,dx$ where $a<b$.

Convenient to define $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

So e.g. we already saw $\int_1^3 x\,dx = 4$. So, $\int_3^1 x\,dx = -4$.

Ex If $\int_1^3 f(x)\,dx = 4$ and $\int_3^7 f(x)\,dx = 16$

what is $\int_1^7 3f(x)\,dx$?

First, $$\int_1^7 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^7 f(x)\,dx$$
$$= 4 + 16 = 20$$

Then, $$\int_1^7 3f(x)\,dx = 3\int_1^7 f(x)\,dx = 3\cdot 20 = \underline{\underline{60}}.$$

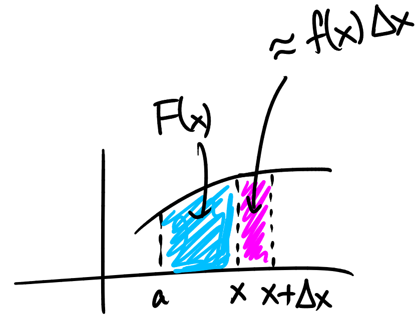
How to actually calculate definite integrals in practice?

Fundamental Theorem of Calculus

Fundamental Theorem of Calculus I:

If $F(x) = \int_a^x f(t)\,dt$ then $F'(x) = f(x)$

[i.e. $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.]



Ex • What is the derivative of

$$F(x) = \int_{-4}^{x} \sin t\,dt \;?$$

By FTC I, $F'(x) = \sin(x)$.

• What is the derivative of

$$F(x) = \int_4^{x^2} \cos t\,dt \;?$$

(Careful — not just $\cos(x^2)$!)

Use chain rule: $\frac{d}{dx}\int_4^{x^2} \cos t\,dt$ $\qquad$ $u = x^2$

$$= \frac{d}{dx}\int_4^{u} \cos t\,dt$$

$$= \underbrace{\frac{du}{dx}}\cdot \underbrace{\frac{d}{du}\int_4^{u} \cos t\,dt}$$

$$= 2x \cdot \cos u$$

$$= 2x \cdot \cos(x^2)$$

Ex $\frac{d}{dx}\int_3^{\ln(1+2x)} \sin(t)\,dt = ?$ $\qquad$ $u = \ln(1+2x)$

$$= \sin(\ln(1+2x)) \cdot \frac{d}{dx} \ln(1+2x)$$

$$= \underline{\underline{\sin(\ln(1+2x)) \cdot \frac{2}{1+2x}}}$$

Ex $\frac{d}{dx}\left(\int_x^5 f(t)\,dt\right) = ?$

Use $\int_x^5 f(t)\,dt = -\int_5^x f(t)\,dt$

So $\frac{d}{dx}\left(\int_x^5 f(t)\,dt\right) = \frac{d}{dx}\left(-\int_5^x f(t)\,dt\right) = -\frac{d}{dx}\int_5^x f(t)\,dt$

$= \underline{\underline{-f(x)}}$ by FTC I

Fundamental Theorem of Calculus II:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$ where $F(x)$ is <u>any</u> antiderivative of $f(x)$.

(<u>Exercise</u>: try to <u>deduce</u> FTC II <u>from</u> FTC I!)

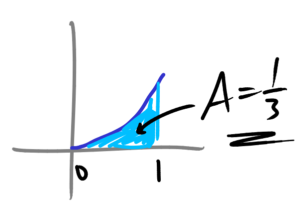
<u>notation</u>: $F(b)-F(a)$ is also written as $F\Big|_a^b$ or $F\Big]_a^b$.

<u>Ex</u>: • Calculate $\int_0^1 x^2\,dx$.

Use FTC II: $F(x) = \frac{1}{3}x^3$ is an antiderivative of x^2, so

$$\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3$$

$$= \frac{1}{3} - 0 = \underline{\underline{\frac{1}{3}}}.$$



• Calculate $\int_0^\pi \sin x\,dx$.

$F(x) = -\cos x$ is an antideriv. of $\sin x$, so

$$\int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = (-\cos \pi) - (-\cos 0)$$

$$= \left(-(-1)\right) - (-1)$$

$$= 1 + 1 = \underline{\underline{2}}$$