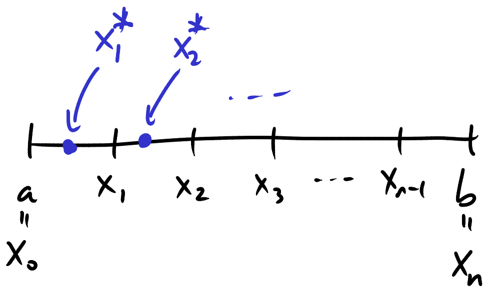
Lecture 19

5 Nov 2015

Last time: <u>definition of definite integral</u>

Given a function f on $[a,b]$

divide $[a,b]$ into n equal subintervals, width $= \Delta x = \frac{b-a}{n}$



$$x_i = x_0 + i\Delta x = a + i\,\frac{b-a}{n}$$

Pick "sample points" x_i^* in the interval $[x_{i-1}, x_i]$

e.g. <u>left endpoint</u>: $x_i^* = x_{i-1}$

<u>right endpoint</u>: $x_i^* = x_i$

Then <u>Riemann sum</u> of f is $\displaystyle\sum_{i=1}^{n} f(x_i^*)\,\Delta x$

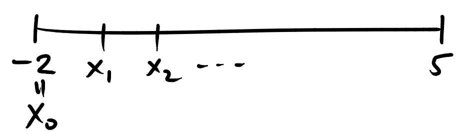
and $$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\,\Delta x\right)$$

<u>Ex</u> Write the definition of $\displaystyle\int_{-2}^{5} e^{2x+1}\,dx$ as a limit, using right endpoints.

$a = -2$

$b = 5$

$$\Delta x = \frac{b-a}{n} = \frac{5-(-2)}{n} = \frac{7}{n}$$



$$x_i^* = x_i = x_0 + i\Delta x = -2 + i\,\frac{7}{n}$$

$f(x) = e^{2x+1}$

Riemann sum: $$\sum_{i=1}^{n} f\left(-2+\frac{7i}{n}\right)\cdot\frac{7}{n}$$

$$= \sum_{i=1}^{n}\left(e^{2\left(-2+\frac{7i}{n}\right)+1}\right)\cdot\frac{7}{n}$$

$$\left(\sum_{i=1}^{n} c\cdot a_i = c\cdot\sum_{i=1}^{n} a_i\right)$$

$$= \frac{7}{n}\sum_{i=1}^{n} e^{-3+\frac{14i}{n}}$$

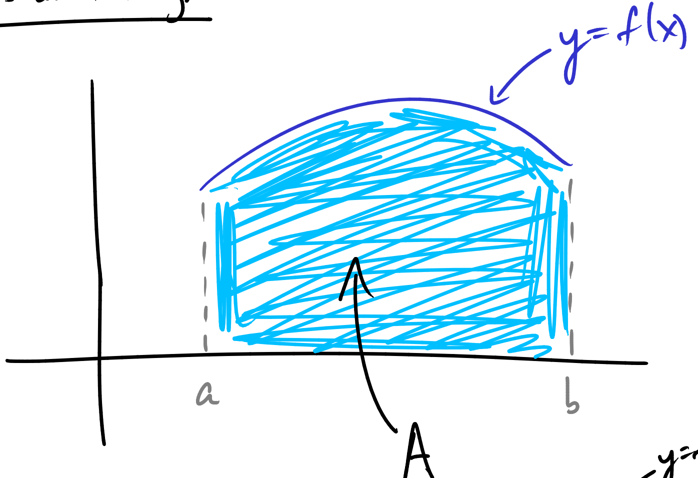
So, $\displaystyle\int_{-2}^{5} e^{2x+1}\,dx = \lim_{n\to\infty}\left(\frac{7}{n}\sum_{i=1}^{n} e^{-3+\frac{14i}{n}}\right)$

(If we used left endpoints, we'd get

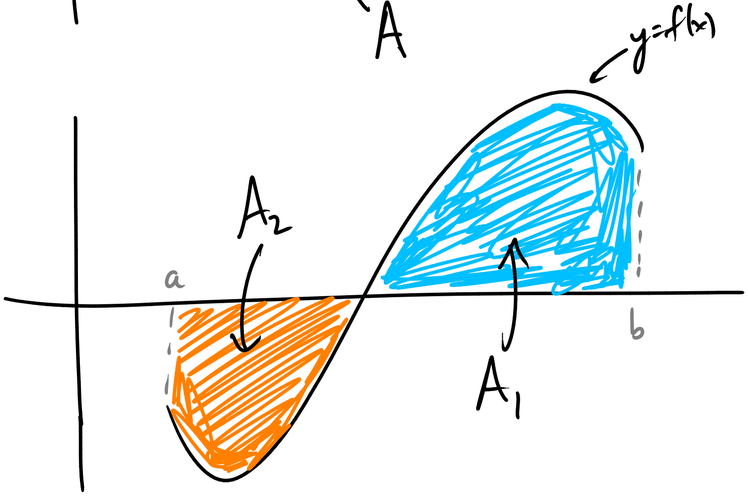
$$\int_{-2}^{5} e^{2x+1}\,dx = \lim_{n\to\infty}\left(\frac{7}{n}\sum_{i=1}^{n} e^{-3+\frac{14(i-1)}{n}}\right)$$

Both are OK!)

Facts about integrals



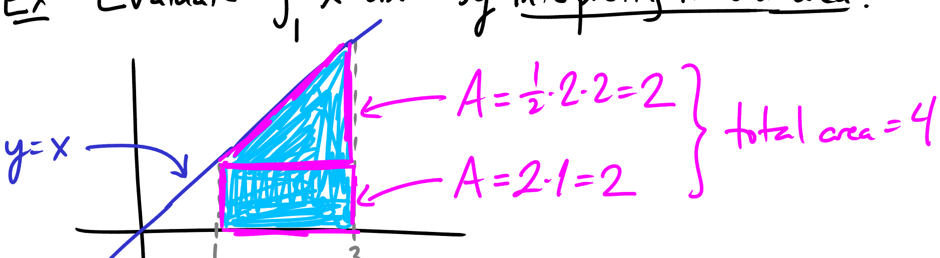
$$\int_a^b f(x)\,dx = A$$



$$\int_a^b f(x)\,dx = A_1 - A_2$$

$$\int_a^b |f(x)|\,dx = A_1 + A_2$$

Ex Evaluate $\displaystyle\int_1^3 x\,dx$ by interpreting it as an area.

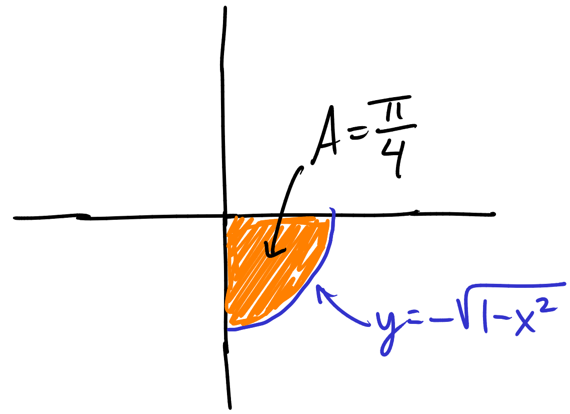


So, $\displaystyle\int_1^3 x\,dx = \underline{\underline{4}}$.

Ex Evaluate $\int_0^1 -\sqrt{1-x^2}\,dx$ by interpreting it as an area.

$y = -\sqrt{1-x^2}$ with x going from 0 to 1.

$y^2 = 1 - x^2$ ie $x^2+y^2=1$ (unit circle)

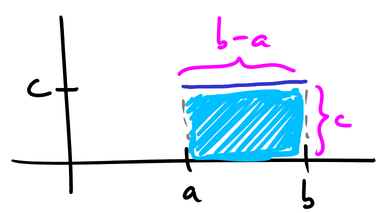


So $\int_0^1 -\sqrt{1-x^2} = -\frac{\pi}{4}$

this $-$ sign because the graph of $y=f(x)$ is below the x-axis!

Basic laws for integrals:

1) $\int_a^b c\,dx = c\cdot(b-a)$

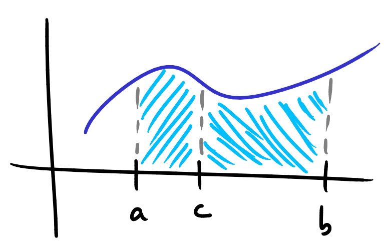


2) $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

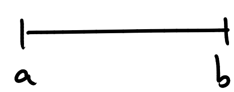
3) $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$

4) $\int_a^b f(x)-g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

5) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



So far, always consider $\int_a^b$ where $b>a$.



Convenient to define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

(e.g. we already saw $\int_1^3 x\,dx = 4$. So, $\int_3^1 x\,dx = -4$.)

$\left[\text{ and } \quad \int_0^1 -\sqrt{1-x^2} = -\frac{\pi}{4}. \quad \text{So, } \int_1^0 -\sqrt{1-x^2} = \frac{\pi}{4}. \right]$

Ex If $\int_1^3 f(x)\,dx = 4$ and $\int_3^7 f(x)\,dx = 16$

what is $\int_1^7 3f(x)\,dx$?

$$\int_1^7 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^7 f(x)\,dx$$

$$= \quad 4 \quad + \quad 16 \quad = 20$$

So, $\int_1^7 3f(x)\,dx = 3 \cdot \int_1^7 f(x)\,dx = 3 \cdot 20 = \underline{60}.$

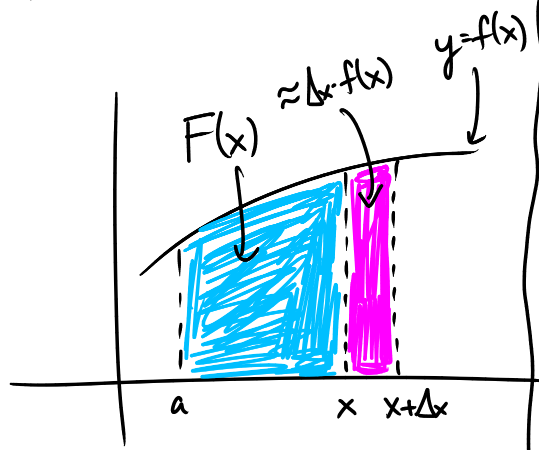
How to actually <u>calculate</u> definite integrals?

<u>Fundamental Theorem of Calculus</u>

Fundamental Theorem of Calculus I:

If $F(x) = \int_a^x f(t)\,dt$ then $F'(x) = f(x)$.

(i.e. $\int_a^x f(t)\,dt$ is an <u>antiderivative</u> of $f(x)$)



$F(x+\Delta x) \approx F(x) + \Delta x \cdot f(x)$

<u>so</u> $F'(x) = f(x)$

Ex What is the derivative of $F(x) = \int_{-4}^x \sin t\,dt$?

$F'(x) = \underline{\sin x}$ by FTC I.

Ex What is the derivative of $F(x) = \int_4^{x^2} \cos t \, dt$?

Use chain rule:

$$\frac{d}{dx} \int_4^{x^2} \cos t \, dt \qquad\qquad u = x^2$$

$$= \frac{d}{dx} \int_4^{u} \cos t \, dt$$

$$= \underbrace{\frac{du}{dx}}_{=2x} \cdot \underbrace{\frac{d}{du} \int_4^{u} \cos t \, dt}_{=\cos u}$$

$$= 2x \cdot \cos u$$

$$= \underline{\underline{2x \cdot \cos(x^2)}}$$

Ex $\frac{d}{dx}\left(\int_x^5 f(t) \, dt \right) = ?$

$$\int_x^5 f(t) \, dt = -\int_5^x f(t) \, dt$$

So $\frac{d}{dx} \int_x^5 f(t) \, dt = \frac{d}{dx}\left(-\int_5^x f(t) \, dt \right) = -\frac{d}{dx} \int_5^x f(t) \, dt = \underline{\underline{-f(x)}}$

by FTC I

Fundamental Theorem of Calculus II:

$\int_a^b f(x) \, dx = F(b) - F(a)$ where $F(x)$ is any antiderivative of $f(x)$.

(Exercise: Deduce FTC II from FTC I!)

notation: $F(b) - F(a)$ is sometimes written $F \Big|_a^b$ or $F \Big]_a^b$.

Ex • Calculate $\int_0^1 x^2 \, dx$.

Use FTC II: $F(x) = \frac{1}{3} x^3$ is an antideriv. of x^2, so

$$\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3$$

$$= \frac{1}{3} - 0 = \frac{1}{3}.$$

- Calculate $\int_0^{\pi} \sin x\,dx$.

  $F(x) = -\cos x$ is an antideriv. of $\sin x$, so

$$\int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} = (-\cos\pi) - (-\cos 0)$$

$$= \big(-(-1)\big) - (-1)$$

$$= 1 + 1 = 2$$